MATH 603: HOMEWORK #2

DUE IN SEBASTIAN MOORE'S MAILBOX BY 4:00 P.M. ON FEB. 8

1. Zeta functions and finitely generated modules

In class we discussed the localization $S^{-1}R$ of a commutative R at a multiplicatively closed set S. By convention, if P is a prime ideal of R, what is meant by the localization R_P of R at the prime P is the localization $(R-P)^{-1}R$. Here $R-P=S$ is a multiplicatively closed set because P is a prime ideal.

1. Let $\mathbb{Z}_{(p)}$ be the localization of the integers at the prime ideal $(p)=\mathbb{Z}p$ generated by a prime p. Show that $\mathbb{Z}_{(p)}$ has Euclidean norm $N:\mathbb{Z}_{(p)}\to\mathbb{Z}_{\geq 0}$ defined by $N(0)=0$ and

   $$N(r/s)=\mathrm{ord}_p(r)-\mathrm{ord}_p(s)=\mathrm{ord}_p(r)$$

   if r and s are non-zero integers, s is prime to p, and $\mathrm{ord}_p(r)$ is the highest power of p dividing r. Show that $\mathrm{Spec}(\mathbb{Z}_{(p)})$ has two prime ideals, $\{0\}$ and the maximal ideal $\mathbb{Z}_{(p)}p$.

2. Show that every finite abelian p-group A is a quotient of $\mathbb{Z}_{(p)}^n$ for some $n\geq 0$. Let $\nu(A)$ be the smallest such n. Show that $\nu(A)$ is the minimal number of generators for A, and that A/pA is a vector space over $\mathbb{Z}/p$ of dimension $\nu(A)$.

3. Let $M=\oplus_{i=1}^n\mathbb{Z}_{(p)}e_i\cong\mathbb{Z}_{(p)}^n$ be a free $\mathbb{Z}_{(p)}$-module of rank $n\geq 1$ on the basis $\{e_i\}_{i=1}^n$. Let $M_1=\mathbb{Z}_{(p)}e_1$, and let $\psi:M\to M_2=M/M_1$ be the natural quotient homomorphism, so that M_2 is free of rank $n-1$. In this problem, all modules are $\mathbb{Z}_{(p)}$-modules. Suppose U is a submodule of finite index in M.

   a. Show that $U_1=U\cap M_1$ has finite p-power index in M_1, and that $\psi(U)=U_2$ has finite p-power index in M_2. Show that U_1 is free of rank 1 over $\mathbb{Z}_{(p)}$ and that U_2 is free of rank $n-1$ over $\mathbb{Z}_{(p)}$. Then show $[M:U]=[M_1:U_1]\cdot[M_2:U_2]$.

   b. Suppose U' is another finite index submodule of M such that

   $$U_1'=U\cap M_1=U_1=\mathbb{Z}_{(p)}f_1 \quad \text{for the element} \quad f_1\in M_1$$

   and

   $$\psi(U')=U_2'=U_2.$$

   Let $\{f_i\}_{i=2}^n$ be a set of elements of U such that $\{\psi(f_i)\}_{i=2}^n$ is a $\mathbb{Z}_{(p)}$ basis for U_2. Show that there exist $h_i\in M_1$ such that $f_i+h_i\in U'$ and $\{f_1\}\cup\{f_i+h_i\}_{i=2}^n$ is a $\mathbb{Z}_{(p)}$ basis for U'. Then show that the residue class $\overline{h_i}$ of h_i in $M_1/U_1=M_1/U_1'$ is determined by U' and the choice of the f_i. Finally, show that for any choice of elements $\{h_i''\}_{i=2}^n\subset M_1$, the submodule U'' generated by $\{f_1\}\cup\{f_i+h_i''\}_{i=2}^n$ has the property that $U''\cap M_1=U_1$ and $\psi(U'')=U_2$.

   c. Show that with the notations of part (a), there are exactly $[M_1:U_1]^{(n-1)}$ distinct submodules U' of M such that $U\cap M_1=M_1$ and $\psi(U)=U_2'=U_2$.

   d. Let $b(n,p^m)$ for $n\geq 1$ and $m\geq 0$ be the number of submodules U of $M=\mathbb{Z}_{(p)}^n$ such that $[M:U]=p^m$. Use part (c) to show

$$b(n,p^m)=\sum_{i=0}^{n}(p^i)^{(n-1)}\cdot b(n-1,p^{m-i}) \tag{1.1}$$

4. A Dirichlet series is a formula sum $\sum_{n=1}^{\infty} a_n n^{-s}$ for some real numbers a_n. We do not require this series to converge for any complex number s; it just records the coefficients a_n. One can add and multiply Dirichlet series in the natural way. With the notations of problem #3, let $\mathcal{U}(M)$ be the set of finite index submodules $U \subset M = \mathbb{Z}_{(p)}^n$.
   a. Show that
   $$\zeta(M,s) = \sum_{U \in \mathcal{U}(M)} [M:U]^{-s}$$
   is a well defined Dirichlet series.
   b. Show that if $M = \mathbb{Z}_p$, so that $n = 1$, one has $\zeta(M,s) = \sum_{i=0}^{\infty} p^{-is} = (1-p^{-s})^{-1}$
   c. Use problem #3 to show by induction that for $n > 1$,
   $$\zeta(\mathbb{Z}_{(p)}^n, s) = \zeta(\mathbb{Z}_{(p)}^{n-1}, s)\left(\sum_{i=0}^{\infty} p^{i(n-1-s)}\right) = \prod_{j=0}^{n-1}(1-p^{j-s})^{-1}.$$

2. More on torsors

Let B be a commutative ring. In class we discussed the group scheme GL_n over $\mathrm{Spec}(B)$. Here $\mathrm{GL}_n = \mathrm{Spec}(A_n(x))$ when $A_n(x) = B[\{x_{i,j}\}, \frac{1}{\det((x_{i,j}))}]$ with $x = \{x_{i,j}\}$ a set of commuting indeterminates corresponding to the entries of an $n \times n$ matrix.

1. Consider the morphism $m : \mathrm{GL}_n \times_{\mathrm{Spec}(B)} \mathrm{GL}_n \to \mathrm{GL}_n$ corresponding to B-algebra map
   $$A_n(x) \to A_n(y) \otimes_B A_n(z)$$
   defined by
   $$x_{i,j} \to \sum_{k=1}^{n} y_{i,k} z_{k,j}.$$
   Show that the morphism m satisfies the natural associative law. Now let C be a commutative B-algebra, and make the natural identifications
   $$\mathrm{GL}_n(C) = Mor_{schemes/B}(\mathrm{Spec}(C), \mathrm{GL}_n) = Hom_{B-algebras}(A_n(x), C) = \mathrm{GL}_n(C).$$
   Show that we have a natural identification
   $$Mor_{schemes/B}(\mathrm{Spec}(C), \mathrm{GL}_n \times_{\mathrm{Spec}(B)} \mathrm{GL}_n) = \mathrm{GL}_n(C) \times \mathrm{GL}_n(C)$$
   and that composing such morphisms with m corresponds to the matrix multiplication map
   $$\mathrm{GL}_n(C) \times \mathrm{GL}_n(C) \to \mathrm{GL}_n(C).$$
2. What should be the identify morphism $e : \mathrm{Spec}(B) \to \mathrm{GL}_n$ and the inverse morphism $u : \mathrm{GL}_n \to \mathrm{GL}_n$ associated to the natural group scheme structure of GL_n? You don't need to verify that these have the right properties.
3. Suppose $n = 2$. How would you define the group $SO(2)$ of 2×2 orthogonal matrices of determinant 1 over B as a subgroup scheme of GL_2?
4. Suppose that B is a field F, and that $c \in F - \{0\}$. Consider the affine scheme
   $$X_b = \mathrm{Spec}(T_b) \quad \text{with} \quad T_b = [x,y]/(x^2+y^2-c).$$
   Show that for all extension fields L of F, the points
   $$X_b(L) = Mor_{schemes}(\mathrm{Spec}(L), X_b)) = Hom_{F-algebras}(T_b, L)$$
   of X_b over L correspond to pairs $(\alpha,\beta) \in L^2$ such that $\alpha^2+\beta^2 = c$.
5. Show that there is an action of $SO(2)$ on X_b over $B = F$ defined by a morphism
   $$SO(2) \times X_b \to X_b$$
   which has the natural effect on points over $L \supset F$, namely it corresonds to the left multiplcation action on $SO(2)(L)$ on column vectors $(\alpha,\beta)^{transpose} \in L^2$ such that $\alpha^2+\beta^2 = c$.

6. Suppose $B = F = \mathbb{R}$ and $c = -1$ in problem # 5. Show that if one base changes to the algebraic closure $\overline{F} = \mathbb{C}$, where is an isomorphism

$$\psi_{\mathbb{C}} : \mathbb{C} \otimes_{\mathbb{R}} X_b = \mathrm{Spec}(\mathbb{C} \otimes_{\mathbb{R}} T_b) \to \mathbb{C} \otimes_{\mathbb{R}} SO(2)$$

which respects the natural left actions of $\mathbb{C} \otimes_{\mathbb{R}} SO(2)$ and which induces the map

$$(\alpha, \beta)^{transpose} \to \begin{array}{|c|c|}\hline \sqrt{-1}\alpha & -\sqrt{-1}\beta \\ \hline \sqrt{-1}\beta & \sqrt{-1}\alpha \\ \hline \end{array}$$

on points over $\mathbb{C}$. Now show that there are no points in $X_b(\mathbb{R})$, and that there is no isomorphism $\psi_F : X_b \to SO(2)$ defined over $F = \mathbb{R}$. This shows X_b is a non-trivial torsor for $SO(2)$ in this case.