MATH 603: HOMEWORK #1

DUE IN SEBASTIAN MOORE'S MAILBOX BY 4:00 P.M. FRIDAY, JAN. 27, 2017

1. P.I.D.'s, Euclidean domains

Suppose R is a Noetherian integral domain with fraction field L.

1. Define an equivalence relation on the set of non-zero finitely generated R-submodules of L in the following way. If M and M' are two such modules, say $M \cong M'$ if and only $M = \lambda \cdot M'$ for some $\lambda \in L^* = L - \{0\}$. Let $[M]$ be the equivalence class of M, and let $Cl(R)$ be the set of all such equivalence classes. Show that $Cl(R)$ has one element if and only if R is a P.I.D.
2. Suppose $n \geq 1$. Define two non-zero n-vectors $c = (c_1, \cdots, c_n)$ and $d = (d_1, \cdots, d_n)$ in L^n to be equivalent if $\lambda \cdot c = d$ for some $\lambda \in L^* = L - \{0\}$. Write $(c_1 : \cdots : c_n)$ for the equivalence class of c. The set of all such $(c_1 : \cdots : c_n)$ is the set $P^{n-1}(L)$ of points of projective $(n-1)$-space over L. The action of $\mathrm{GL}_n(L)$ on column vectors gives an action of $\mathrm{GL}_n(L)$ on $P^{n-1}(L)$.
   a. Show that there is a well defined map $\psi : P^{n-1}(L) \to Cl(R)$ which sends $(c_1 : \cdots : c_n)$ to $[Rc_1 + \cdots + Rc_n]$. Show that this map is surjective if and only if every R-ideal is generated by at most n elements.
   b. Show that the left action of the group $\mathrm{GL}_n(R)$ on column vectors gives an action of $\mathrm{GL}_n(R)$ on $P^{n-1}(L)$. Show that the map ψ in part (a) is constant on the orbits of this action, so that it gives a map $\overline{\psi} : \mathrm{GL}_n(R)\backslash P^{n-1}(L) \to Cl(R)$.
3. Suppose R is Euclidean with respect to some norm $N : R \to \mathbb{Z}_{\geq 0}$. How would you use the Euclidean algorithm to find a matrix $M \in \mathrm{SL}_2(R)$ which moves a given point $(a : b) \in P^2(L)$ to $(1 : 0)$? (Keep in mind that a and b are in L but may not be in R.)

Remarks: Suppose $R = R_d$ is one of the quadratic integer rings discussed in class with $d < 0$ a square-free integer. Then in fact, $CL(r_d)$ is a finite set, and its order is the class number $h(R_d)$ of R_d. The Gauss class number problem discussed in class was to show that there are exactly 9 such d for which this number is 1. By problem #1, the latter condition is equivalent to R_d being a P.I.D.. The result of Gross and Zagier and Goldfeld gave for the first time an *effective* lower bound for $h(R_d)$ which goes to infinity with $|d|$.

2. An example of torsors

Fix $n = 2$ in problem #2. Suppose that $c = (c_1, c_2)$ and $d = (d_1, d_2)$ are in $L^2 - \{0, 0\}$ and that $\psi((c_1 : c_2)) = \psi((d_1 : d_2))$. In view of problem #2(b), it's a natural question whether $(c_1 : c_2)$ and $(d_1 : d_2)$ are in the same orbit under the action of $\mathrm{GL}_2(R)$, or even under the action of $\mathrm{SL}_2(R)$. This problem connects this question to the study of torsors for algebraic groups. Let c^T and d^T be the column vectors which are the transposes of c and d.

4. Show that after replacing d by a $\lambda \cdot d$ for some $\lambda \in L^*$, one can assume $Rc_1 + Rc_2 = Rd_1 + Rd_2$.
5. Show that there is always a matrix $m \in \mathrm{GL}_2(L)$ such that $mc^T = d^T$. Then show that for each $\alpha \in L^*$, there is a matrix $z \in \mathrm{GL}_2(L)$ so $\det(z) = \alpha$ and $zc^T = c^T$. Finally, show that the set $S = \{m \in \mathrm{SL}_2(L) : mc^T = d^T\}$ consists of one orbit under the left multiplication action of the subgroup $H = \{\gamma \in \mathrm{SL}_2(L) : \gamma d^T = d^T\} \subset \mathrm{GL}_2(L)$.

6. Prove that H is conjugate in $\mathrm{SL}_2(L)$ to the subgroup of upper triangular matrices with 1 down the diagonal. (It is useful to first consider the action of $\mathrm{SL}_2(L)$ on $L^2 - \{(0,0)\}$.)
7. Show that if

$$m = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \tag{2.1}$$

is a matrix of indeterminants, then there is a finite set of polynomials $P \subset L[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]$ such that m defines an element of H if and only if every polynomial in P vanishes at $x = (x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2})$. (This shows H is a linear algebraic group.)

8. Show that similarly, there is another finite set of polynomials $P' \subset L[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]$ such that m is in S if and only if every polynomial P' vanishes at x. (This shows S is a torsor for H, since S consists of one H orbit.)
9. Finally, show that there is a matrix $m \in \mathrm{SL}_2(R)$ such that $mc^T = d^T$ if and only if the polynomials in P' vanish at some $x \in R^4$. (One then says that the H-torsor S has a point over R.)
10. Prove that if $c = (1,0) = (c_1, c_2)$ and $Rc_1 + Rc_2 = Rd_1 + Rd_2 = R$ there is always a matrix $m \in \mathrm{SL}_2(R)$ such that $mc^T = d^T$. Deduce from this that the map $\overline{\psi} : \mathrm{GL}_2(R)\backslash P^{n-1}(L) \to Cl(R)$ in problem #2(b) has the property that the inverse image of the class $[R]$ of $Cl(R)$ has exactly one element.

Remarks: Problem #9 illustrates the problem of finding whether or not a torsor for a linear algebraic group has a point over a given ring. We will return later in the semester for some necessary and sufficient conditions for this to be true. This is a very active research area.