MATH 350: HOMEWORK #3

DUE IN LECTURE FRIDAY, OCT. 3, 2014.

1. G.C.D.'s

1. Write the g.c.d. of 666 and 1414 as an integral combination of 666 and 1414.
2. The improved division algorithm states that given integers a and $b \neq 0$, there are always integers q and r such that $a = qb + r$ and $|r| \leq |b|/2$. In class we discussed improving the Euclidean algorithm by using the improved division algorithm. Suppose now that $a \geq b$ and that $b > 0$ has at most k decimal digits, so that $0 < b < 10^k$. Show that the number of divisions which are needed using the improved Euclidean algorithm is bounded above by $(k/\log_{10}(2)) + 1$.

2. The fundamental theorem of arithmetic

3. Which positive integers have exactly three positive divisors? Which have exactly four positive divisors?
4. Suppose r is a real number. Define $\lfloor r \rfloor$ to be the largest integer which is less than or equal to r. Then $\lfloor r \rfloor$ is called the floor function of r. Show that if $n \geq 1$ is an integer and p is a prime, then the power of p which divides $n!$ is
$$\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \cdots$$
Use this to write down the prime factorization of 13!.
5. Let H be the set of all positive integers of the form $4k + 1$ for some integer $k \geq 0$. Say an element $h \in H$ is a Hilbert prime if the only ways to factor h into a product of elements of h is by writing $h = 1 \cdot h$ and $h = h \cdot 1$. Show that every element of H can be factored into a finite product of Hilbert primes. Then show this factorization need not be unique by finding two different factorizations of 693.

3. Diophantine approximation and irrational numbers

6. Show that if $a, b, c, d \in \mathbb{Z}$ with $b \neq 0 \neq d$, and the rational numbers a/b and c/d are distinct, then
$$|\frac{a}{b} - \frac{c}{d}| \geq \frac{1}{|bd|}.$$
7. Use problem # 6 to show that
$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots$$
is irrational. (Hint: Suppose $e = a/b$ is rational, and consider $c/N! = \sum_{n=0}^{N} \frac{1}{n!}$ for large N.)