MATH 350: HOMEWORK #2

DUE IN LECTURE FRIDAY, SEPT. 19, 2014.

1. Ordinal numbers

Recall that if $(B, \leq)$ is a well-ordered set, the Von Neumann function $f : B \to \text{Sets}$ is characterized by

$$f(m) = \{f(m') : m' \in B \quad \text{and} \quad m' < m\}$$

We showed in class that f gives an order preserving bijection $f : B \to f(B) = \{f(m) : m \in B\}$. The set $f(B)$ is the Von Neumann ordinal number associated to B. It is determined by, and determines, the order type of $(B, \leq)$. Let $f' : B' \to f(B')$ be the Von Neumann function associated to another well ordered set $(B', \leq')$.

1. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the natural numbers with the usual ordering. The ordinal ω is $f(\mathbb{N})$. Describe an order preserving injection $h : \mathbb{N} \to \mathbb{R}$ such that $\sum_{i=1}^{\infty} h(i)$ converges in the sense of one variable calculus. Thus $h(\mathbb{N})$ is a well-ordered subset of $\mathbb{R}$ with respect to the usual ordering $\leq$ which has order type ω and the sum of the elements of $h(\mathbb{N})$ converges.
2. We defined the sum $f(B) + f'(B')$ to be $f''(B'')$ when $f'' : B'' \to f(B'')$ is the Von Neumann function of the well ordered set B'' which consists of B followed by B''. Show that the ordinal $\omega + \omega$ is countable, and describe a subset $\{a_i\}_{i \in \omega+\omega}$ of the real numbers which has order type $\omega + \omega$ with respect to the usual ordering $\leq$ of $\mathbb{R}$.

   Extra Credit: Can you find a bijection $h : \mathbb{N} \to \omega + \omega$ and a set $\{a_i\}_{i \in \omega+\omega}$ as above such that $\sum_{i=1}^{\infty} a_{h(i)}$ converges in the sense of one variable calculus?
3. The product of two ordinals $f(B)$ and $f'(B')$ is defined to be $f_0(B_0)$ when f_0 is the Von Neumann function associated to the product set $B_0 = B \times B'$ with the following lexicographic order. If (b, b') and (b_0, b'_0) are in $B \times B'$, then $(b, b') \leq (b_0, b'_0)$ if either $b' < b'_0$, or $b' = b'_0$ and $b \leq b_0$. With $\omega = f(\mathbb{N})$ as in problem # 1, describe a set $\{a_i\}_{i \in \omega \times \omega}$ of real numbers which has order type $\omega \times \omega$ with respect to the usual order $\leq$ of the real numbers.
4. Suppose $(B, \leq)$ is already an ordinal number, i.e. $B = f'(B')$ for some well ordered set $(B', \leq')$. Show that $f(B) = B$. (You can use that an ordinal number is determined by its order type.)
5. Show that for any two well ordered sets $(B, \leq)$ and $(B', \leq)$, one has either $f(B) \subset f'(B')$ or $f'(B') \subset f(B)$.

   (Hints: To get a contradiction, suppose $f(B)$ is not a subset of $f'(B')$ and that $f'(B')$ is not a subset of $f(B)$. Show that there are then minimal elements m and m' of B and B', respectively, such that $f(m)$ is not in $f'(B')$ and $f'(m')$ is not in $f(B)$. Here $m_0 < m$ implies that $f(m_0) \in f'(B')$, so that $f(m_0) = f'(m'_0)$ for some unique $m'_0 \in B'$. Using the definitions of f and f', show that it is impossible that $m'_0 = m'$ or $m'_0 > m'$, so that in fact $m'_0 < m'$. Apply this argument with the roles of B and B' reversed to show that $\{f(m_0) : m_0 < m \quad \text{in} \quad B\}$ and $\{f'(m'_0) : m'_0 < m' \quad \text{in} \quad B'\}$ have to be equal. Use this to show $f(m) = f'(m_0)$, and explain why this is a contradiction.)