MATH 210, PROBLEM SET 7

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1. Poisson processes and rate constants

The object of these problems is to relate the rate constants appearing in the simplest epidemic model we've discussed to hypotheses about how people develop and recover from diseases.

As in the notes, let $S(t)$, $I(t)$ and $R(t)$ be the proportion of the total population consisting of susceptibles, infected individuals and individuals who have been removed. Then

$$S(t) + I(t) + R(t) = 1$$

and $S(t)$ and $I(t)$ satisfy differential equations

$$\frac{dS}{dt} = -\alpha S \cdot I \quad \text{and} \quad \frac{dI}{dt} = \alpha SI - \delta_2 I \tag{1.1}$$

for some constants $\alpha, \delta_2 \geq 0$. The term αSI represents the increase in the infected population resulting from contact with susceptibles, while the term $\delta_2 I$ represents the rate at which infected individual are becoming removed due to recovery or passing away.

Problem 1 If $\alpha = 0$, then (1.1) gives

$$\frac{dI}{dt} = -\delta_2 I.$$

This corresponds to infected individuals not being contagious but becoming removed at a rate δ_2. Explain why the solution

$$I(t) = I(0)e^{-\delta_2 t} \tag{1.2}$$

in this case can be interpreted as saying that the odds that a person who is infected at time 0 will still be infected at time t are equal to $e^{-\delta_2 t}$. Recall that if X is a Poisson random variable associated to a process which produces events at a rate of δ_2 events per unit time, the probability of observing j events in time t is

$$Prob(X = j) = e^{-\delta_2 t}\frac{(\delta_2 t)^j}{j!}. \tag{1.3}$$

Explain, using $j = 0$ in (1.3), why the formula (1.2) makes sense if we imagine that removal events happen to an infected person according to a Poisson process at the rate of δ_2 per unit time, and the first removal event leads to an infected person becoming removed.

Problem 2 Problem #1 explains that δ_2 is the rate per unit time associated to a Poisson process X which represents the number of removal events which occur in time t to an infected person. Let Y be the random variable giving the first time a removal event occurs. Explain why

$$Prob(Y > t) = Prob(X = 0) = e^{-\delta_2 t}.$$

We are assuming that events occur at a certain rate per unit time. Explain why this implies $Prob(Y = t) = 0$ using that the one point set $\{t\}$ represents a time interval of length 0. So

$$Prob(Y \geq t) = Prob(Y > t) = e^{-\delta_2 t} = \int_t^\infty f_Y(r)dr$$

when $f_Y(r)$ is the density function of Y. Show that this implies

$$\delta_2 \cdot e^{-\delta_2 t} = f_Y(t).$$

Such Y are called exponentially distributed random variables with parameter δ_2. Show that the expected value $T = E(Y)$ of Y is $1/\delta_2$ using integration by parts. Finally explain why this means that

$$\delta_2 = 1/T \tag{1.4}$$

when T is the expected value of the time it takes for an infected person to be removed.

Problem 3 Suppose now that $\delta_2 = 0$ in the model (1.1), so that (1.1) gives

$$\frac{dS}{dt} = -\alpha SI \quad \text{and} \quad \frac{dI}{dt} = \alpha SI. \tag{1.5}$$

and assume $\alpha > 0$. Let T be as in problem 2 the expected time it takes for an infected person to be removed. An extra credit problem below is to show that if $\epsilon > 0$ is an arbitrary small positive constant, there is a positive constant $\delta > 0$ depending on ϵ such that if $S(0) > 1 - \delta$ then

$$1 \geq S(0) \geq S(t) \geq S(T) \geq 1 - \epsilon \quad \text{for} \quad 0 \leq t \leq T. \tag{1.6}$$

Show, using (1.5) and (1.6) and $S(t) + I(t) + R(t) = 1$, that

$$(1 - \epsilon) \cdot \alpha \leq \frac{1}{I} \cdot \frac{dI}{dt} = \frac{d\ln(I)}{dt} \leq \alpha \quad \text{for} \quad 0 \leq t \leq T. \tag{1.7}$$

Deduce from this that if $S(0) > 1 - \delta$ then

$$e^{(1-\epsilon)\alpha t} I(0) \leq I(t) \leq e^{\alpha t} I(0) \quad \text{for} \quad 0 \leq t \leq T. \tag{1.8}$$

Problem 4 Show that (1.8) is compatible with the following model for how new infections occur. Suppose that the initially susceptible population $S(0)$ is very nearly the entire population, i.e. $S(0) > 1 - \delta$, and that $I(0) > 0$ so that $I(0) \leq \delta$ since $S(0)+I(0) = 1-R(0) \leq 1$. Suppose $\delta_2 = 0$, so that we consider only new infections. In each small time interval Δ, contact between an infected person and the entire susceptible population has a probability of $\alpha\Delta$ of producing a new infection. If Δ is small enough, at most one new infection of this kind is produced in each time interval. What happens in different time intervals does not depend on what happens in other time intervals. Show that (1.8) is consistent with this model. Namely, let $\Delta \to 0$, and explain why the limit

$$\lim_{m\to\infty} I(0) \cdot (1 + \alpha \cdot \frac{t}{m})^m \tag{1.9}$$

should be $I(t)$. Then evaluate

$$\lim_{m\to\infty} m \cdot \ln(1 + \alpha \cdot \frac{t}{m}) \tag{1.10}$$

using L'Hopital's rule and explain why (1.9) is consistent with (1.8) as we let $\epsilon \to 0^+$. Explain why this leads to interpreting α as the rate per unit time that an infected person produces new infections when almost everyone is susceptible and we ignore recoveries from an infection.

Comment: The constant $\delta_2/\alpha = 1/(\alpha T)$ is the reciprocal of the number of new infections an infected person is expected to produce over time interval T in a population which is almost entirely susceptible. The notes show that when

$$S(0) > 1/(\alpha T)$$

and $I(0)$ is near 0, the number if infections initially rises, corresponding to an epidemic. On the other hand, if

$$S(0) < 1/(\alpha T)$$

and $I(0)$ is near 0, $I(t)$ decreases monotonically to 0. Thus $1/(\alpha T)$ represents the maximal fraction of the total population which can be susceptible if herd immunity is to occur. Suppose, for example, that 20% of the population of New York City has antibodies demonstrating that they are no longer susceptible to covid-19. Then $S(0) = 0.80 = 1/(1.25)$, so herd immunity occurs in this case when the number of new infections produced over the time T it takes a person to recover or pass away is $\alpha T = 1.25$. Without social distancing, earlier estimates had $\alpha T \geq 2.5$. So to achieve herd immunity when $S(0) = 0.80$, one must practice at least enough social distancing to cut by a factor of 2 the number of new infections produced per infected person.

Problem 5 **(Extra Credit!)** This problem has to do with showing (1.6). Assume as in problem 3 that $\alpha > 0$, $\delta_2 = 0$ and $I(0) > 0$. Use (1.1) to show

$$I(0) \leq I(t) \leq I(T) \leq I(0)e^{\alpha T} \quad \text{for} \quad 0 \leq t \leq T.$$

Deduce from this that

$$0 \geq \frac{dS}{dt}(t) \geq -\alpha S I(0) e^{\alpha T} \quad \text{for} \quad 0 \leq t \leq T. \tag{1.11}$$

and then that

$$S(0) \geq S(t) \geq S(T) \geq S(0)e^{-T\alpha I(0)e^{\alpha T}} \quad \text{for} \quad 0 \leq t \leq T. \tag{1.12}$$

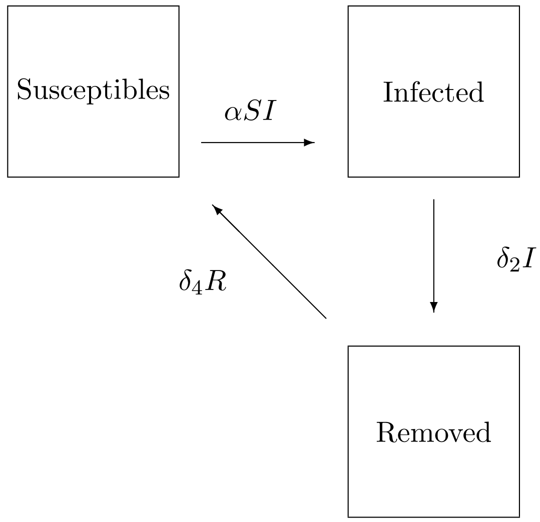
Suppose $\epsilon > 0$ is an arbitrary small positive constant. Show that there is a sufficiently small positive constant δ, depending on ϵ, such that if $S(0) \geq 1 - \delta$ then $0 < I(0) \leq \delta$ and

$$1 \geq (1-\delta) \cdot e^{-T\alpha I(0)e^{\alpha T}} \geq 1 - \epsilon. \tag{1.13}$$

Then use (1.12) and (1.13) to show (1.6).

2. Equilibria for a model in which removed individuals can become susceptible again

In this section we'll work out the linear stabilitiy analysis for the equilibria associated to a variation on the model in the notes. The variation has the following diagram:



In this model, people who have become removed, e.g. by recovering from an infection, return to being susceptible at a certain rate δ_4. This model is important to consider, since there is some evidence that people who have recovered from covid-19 and acquired some immunity as a result can lose that immunity over time. We will assume that α, δ_2 and δ_4 are all positive.

The differential equations are

$$\frac{dS}{dt} = -\alpha SI + \delta_4 R \tag{2.14}$$

$$\frac{dI}{dt} = \alpha SI - \delta_2 I \tag{2.15}$$

$$\frac{dR}{dt} = \delta_2 I - \delta_4 R \tag{2.16}$$

Problem 6 Show that $S + I + R$ does not vary with time. We will normalize this sum so $S(t) + I(t) + R(t) = 1$, corresponding as before to these functions representing the proportion of the total population which is susceptible, infected and removed.

Problem 7 To simplify the calculations, recall from problem #2 that δ_2 can be interpreted as $1/T$ when T is the expected time it takes for an infected person to become removed. Let's choose our unit of time so that $T = 1$, leading to $\delta_2 = 1$. Use Problem #6 to show that $S(t)$ and $I(t)$ now satisfy the system of differential equations

$$\frac{dS}{dt} = -\alpha SI + \delta_4(1 - S - I) = G_1(S, I) \tag{2.17}$$

$$\frac{dI}{dt} = \alpha SI - I = G_2(S, I) \tag{2.18}$$

Find all values for $(S(0), I(0))$ which are physically meaningful equilibria of this system. Recall that we are assuming $\alpha > 0$ and $\delta_4 > 0$. The equilibria should be physically meaningful in the sense that $S(0)$, $I(0)$ and $R(0) = 1 - S(0) - I(0)$ all lie in the interval $[0, 1]$ because they represent non-negative proportions of the entire population.

Problem 7 Calculate the value of the Jacobian matrix

$$Jac(G) = \begin{pmatrix} \frac{\partial G_1}{\partial S} & \frac{\partial G_1}{\partial I} \\ \\ \frac{\partial G_2}{\partial S} & \frac{\partial G_2}{\partial I} \end{pmatrix} \tag{2.19}$$

at each of the equilibria you found in Problem # 6.

Problem 8 Use problem #7 to determine for each of the equilibria you found in Problem # 6, which values of α and δ_4 indicate that the equilibrium is linearly stable. You can use the fact that a two-by-two matrix with real entries has eigenvalues with negative real parts if and only if it has a positive determinant and a negative trace.