MATH 210, PROBLEM SET 6, ADDITIONAL EXTRA CREDIT

THESE PROBLEMS ARE OPTIONAL. SEND YOUR ANSWERS BY E-MAIL TO HAO ZHANG BY 5 P.M. MAY 7 FOR EXTRA CREDIT

Email of Hao Zhang: zhangphy@sas.upenn.edu

1. More on problem 3 of Homework 6

During class we discussed the formula

$$Prob(T_z = i | X_1 = 1) = Prob(T_{z+1} = i - 1) \quad \text{for} \quad 0 < z < N \tag{1.1}$$

when T_z is the random variable giving the time a random walk starting at the integer z takes to reach either 0 or N, and X_1 is the random variable with values in $\{\pm 1\}$ which gives the first step in the walk. For the purpose of writing up homework 6, it's acceptable to argue that a walk of length i starting at z with first step $X_1 = 1$ lands after the first step at $z + 1$ and must then continue for $i - 1$ steps in order to first reach 0 or N at the i^{th} step. It's enough to explain why (1.1) follows from this by interpreting conditional probability as the probability that something will happen given that you know something else has happened.

This sort of argument is common but not rigorous. It doesn't deduce (1.1) from our definition of events as subsets of a sample space. A careful proof ultimately does require a counting argument. This set of extra credit problems is designed to produce a rigorous proof of (1.1).

1. Recall from problem #1 of homework 6, the natural sample space to use is the set S of all infinite sequences $s = (s_1, s_2, \ldots, \ldots)$ of integers $s_i \in \{\pm 1\}$, with $X_i(s) = s_i$ being the i^{th} step in a random walk beginning at time 0 at position z. Show that the event

$$L(z, i) = \{s \in S : T_z(s) = i\}$$

can be described in the following way for $i \geq 0$. Let $A(z, i)$ be the set of all ordered i-tuples $a = (a_1, \ldots, a_i)$ of elements $a_j \in \{\pm 1\}$ such that

$$z + \sum_{j=0}^{i} a_j \in \{0, N\} \quad \text{and} \quad 0 < z + \sum_{j=0}^{\ell} a_j < N \quad \text{for} \quad 0 \leq j < i.$$

For $a = (a_1, \ldots, a_i) \in A(z, i)$ define the event

$$U(a) = \{s = (s_1, s_2, \ldots) \in S : s_j = a_j \quad \text{for} \quad 1 \leq j \leq i\}.$$

In other words,

$$U(a) = \{s \in S : X_1(s) = a_1, X_2(s) = a_2, \ldots, X_i(s) = a_i\}. \tag{1.2}$$

Show that

$$L(z, i) = \{s \in S : T_z(s) = i\} = \bigcup_{a \in A(z,i)} U(a). \tag{1.3}$$

2. Show that the union on the right in (1.3) is disjoint, in the sense that if $a, a' \in A(z,i)$ and $a \neq a'$ then $U(a) \cap U(a') = \emptyset$. Show that the fact that $X_1, \ldots, X_i$ represent independent coin flips implies via (1.2) that

$$Prob(U(a)) = 2^{-i} \quad \text{for all} \quad a \in A(z,i). \tag{1.4}$$

Conclude that

$$Prob(T_z = i) = Prob(L(z,i)) = \sum_{a \in A(z,i)} Prob(U(a)) = \#A(z,i) \cdot 2^{-i}. \tag{1.5}$$

3. For $a' = (a'_1, \ldots, a'_{i-1}) \in A(z+1, i-1)$, show that the i-tuple

$$\alpha(a') = (1, a'_1, \ldots, a'_{i-1})$$

lies in $A(z,i)$ and conversely that if $a = (a_1, \ldots, a_i) \in A(z,i)$ and $a_1 = 1$, then $a = \alpha(a')$ for a unique $a' \in A(z+1, i-1)$. Use this to show that the event

$$\{s \in S : X_1(s) = 1 \quad \text{and} \quad T_z(s) = i\}$$

equals the disjoint union

$$\bigcup_{a' \in A(z+1, i-1)} U(\alpha(a')).$$

Conclude that

$$Prob(X_1 = 1 \quad \text{and} \quad T_z = i) = \sum_{a' \in A(z+1,i-1)} Prob(U(\alpha(a'))) = \#A(z+1, i-1) \cdot 2^{-i}. \tag{1.6}$$

4. Show that if we replace z by $z+1$ and i by $i-1$ in (1.5), we get

$$Prob(T_{z+1} = i-1) = \#A(z+1, i-1) \cdot 2^{-i+1}.$$

Use the definition of conditional probability to show

$$Prob(T_z = i | X_1 = 1) = 2 \cdot Prob(X_1 = 1 \quad \text{and} \quad T_z = i)$$

Then use these results and (1.6) to finally check (1.1).