LINEAR PROGRAMMING PROBLEMS

MATH 210 NOTES

1. Statement of linear programming problems

Suppose $n, m \geq 1$ are integers and that

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,m} \\ a_{i,1} & a_{i,j} & a_{i,m} \\ a_{n,1} & \dots & a_{n,m} \end{pmatrix}$$

is an $n \times m$ matrix of **positive** real constants. Let

$$b = (b_1, \dots, b_m) \quad \text{and} \quad c = (c_1, \dots, c_n)$$

be vectors with **positive** entries.

The linear programming problem associated to A, b and c is to find all vectors

$$s = (s_1, \dots, s_n) \geq (0, \dots, 0)$$

such that

$$sA \geq b$$

and for which the real number

$$f(s) = c_1 s_1 + \dots + c_n s_n = sc^{Transpose}$$

is minimized. Here an inequality

$$(d_1, \dots, d_\ell) \geq (e_1, \dots, e_\ell)$$

between two vectors of the same length means $d_i \geq e_i$ for all i.

2. Two ways that linear programming problems arise

This section should be omitted from Friday's lecture - it's just for background.

2.1. Optimal resource allocation.

Suppose $s = (s_1, \dots, s_n)$ represents the different amounts of n-different resources which can be used to manufacture m different kinds of products. The entries of the row vector sA could represent the number of each kind of product which can be produced using the resources represented by s. So $s \geq (0, \dots, 0)$ means that we have to use non-negative amounts of each resources, and the constraint $sA \geq b$ sets a lower limit to the number of each kind of product which must be produced. The function $f(s) = c_1 s_1 + \dots + c_n s_n$ could represent the cost of using these resources. So linear programming amounts to finding all optimal choices for the resources to be used which can be used to reach certain target numbers of each kind of product while minimizing cost.

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2.2. **Two person zero sum games.** Another way linear programming arises if from two-person zero sum games. Suppose that

$$B = \begin{pmatrix} b_{1,1} & \dots & b_{1,m} \\ b_{i,1} & b_{i,j} & b_{i,m} \\ b_{n,1} & \dots & b_{n,m} \end{pmatrix}$$

represents the payoff matrix to player 1 in a two person zero sum game in which player 1 has n strategies and player 2 has m strategies. If the players play their options with probability vectors represented by $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_m)$ then the expected payoff to player 1 is

$$E(p, q) = pBq^{Transpose}$$

since the payoff to player 1 for playing option i when player 2 plays option j is $b_{i,j}$. Player 1 wishes to find a probability vector $p^* = (p_1, \dots, p_n)$ so

$$\min_q E(p^*, q) = \max_p(\min_q E(p, q))$$

where p and q range over all probability vectors of the correct size. This is because for each choice of p, player 2 will look for the q which minimizes the expected payoff to player 1. So player 1 wants to maximize their worst case outcome.

The main theorem relating p^* to linear programming is this. Choose $\delta > 0$ so that the matrix A produced by adding δ to all entries of B has only positive entries. Let

$$b = (1, \dots, 1) \quad c = (1, \dots, 1)$$

be vectors of length m and n respectively in the linear programming problem associated to A. Then the map

$$s \to s/f(s) = p^*$$

defines a bijection between solutions to the linear programming problem associated to A, b, c as above and optimal strategies for player 1 in the game theory problem associated to B.

3. Proof that linear programming problems have solutions

Lemma 3.1. *There is at least vector $\hat{s} = (\hat{s}_1, \dots, \hat{s}_n)$ satisfying $\hat{s} \geq (0, \dots, 0)$ and $\hat{s}A \geq b$.*

Proof. Since the entries of A are assumed to be positive, just take $\hat{s}$ to have sufficiently large entries. □

Lemma 3.2. *The set of T vectors $s = (s_1, \dots, s_n)$ such that $s \geq (0, \dots, 0)$, $sA \geq b$ and $f(s) \leq f(\hat{s})$ is a closed bounded set.*

Proof. The students have taken math 114. So in principle, they should know what open and closed sets are, but one should review this. I would recommend first defining an open subset U of $\mathbb{R}^n$ to be a set which is either empty or such that all $u \in U$ are contained in some open ball which is entirely contained in U. Then draw a picture, and make the precise definition of a ball of radius $r > 0$ around u in $\mathbb{R}^n$. A closed set is then the complement of an open set. To show T is closed, one needs to show $\mathbb{R}^n - T = U$ is closed. Here $u \in U = \mathbb{R}^n - T$ if and only if one of the inequalities $u \geq (0, \dots, 0)$, $uA \geq b$ or $f(u) \leq f(\hat{s})$ is not true. The inequality which fails says either that

$$d_1u_1 + \dots d_nu_n < e$$

for some constants d_i and e (in the event it is one of the inequalities represented by $u \geq (0, \dots, 0)$ or $uA \geq b$) or that

$$f(u) = c_1u_1 + \dots + c_nu_n > f(\hat{s}).$$

For all $u' = (u'_1, \ldots, u'_n)$ in a sufficiently small all around u, the same inequality will fail. So U open and T is closed. To show T is bounded, recall that $s \in T$ implies $s = (s_1, \ldots, s_n) \geq (0, \ldots, 0)$ and $f(s) = c_1 s_1 + \ldots + c_n s_n \leq f(\hat{s})$ where the c_i have been assumed to be positive. This puts an upper and lower bound on each s_i. □

Now we quote a Theorem from real analysis:

Theorem 3.3. *A continuous real valued function on a non-empty closed bounded subset of $\mathbb{R}^n$ assumes its minimum value on the set.*

Here one may need to remind people what a continuous function is, and that linear functions are continuous.

Now one completes the proof of the existences of solutions to linear programming problems by saying that the solutions s are the elements of the non-empty closed bounded set T where $f(s)$ takes on its minimum value.

4. The vertex method for solving linear programming problems

There are $n + m$ linear inequalities which define a linear progamming problem are contained in the requirements that

$$s = (s_1, \ldots, s_n) \geq (0, \ldots, 0)$$
$$\text{and} \quad sA \geq b = (b_1, \ldots, b_m).$$

These can be written in the form

$$\begin{gathered} T_1 : s_1 \geq 0 \\ T_2 : s_2 \geq 0 \\ \cdots \\ T_n : s_n \geq 0 \\ T_{n+1} : s_1 a_{1,1,} + s_2 a_{2,1} + \cdots s_n a_{n,1} \geq b_1 \\ \cdots \\ T_{n+m} : s_1 a_{1,m,} + s_2 a_{2,m} + \cdots s_n a_{n,m} \geq b_m \end{gathered}$$

Write the q^{th} of these inequalities as

$$T_q : h_q(s) \geq d_q$$

where $h_q(s)$ is a linear function of s and d_q is a real number.

Definition 4.1. For $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ let $T(s)$ be the set of all the inequalities T_q which hold with equality at s. A point s is a vertex of the linear programming problem if

1. s is the unique point in $\mathbb{R}^n$ where the inequalities in $T(s)$ hold with equality, and
2. All of the $n + m$ inequalities defining the linear programming problem hold at s.

Let $\mathcal{V}$ be the set of all vertices. Define the subset $\mathcal{O}$ of optimal vertices to be

$$\mathcal{O} = \{v \in \mathcal{V} : f(v) \leq f(v') \text{ for all } v' \in \mathcal{V}\}$$

where $f(s) = c_1 s_1 + \cdots + c_n s_n$ is the function to be minimized.

Definition 4.2. Suppose $C = \{c(1), \ldots, c(\ell)\}$ is a non-empty set of vectors in $\mathbb{R}^n$. The span $\text{Span}(C)$ of C is defined to be the set of all linear combinations

$$r_1 c(1) + \cdots r_\ell c(\ell)$$

for which $r_1, \ldots, r_\ell \in \mathbb{R}$, $r_i \geq 0$ for all i and $r_1 + \cdots + r_\ell = 1$.

We will give a proof of the following result.

Theorem 4.3. *The set $\mathcal{O}$ of optimal vertices is non-empty, and the set $\mathcal{S}$ of all solutions s of the linear programming is the span* $\mathrm{Span}(\mathcal{O})$ *of* $\mathcal{O}$.

Lemma 4.4. *Suppose $s \in \mathcal{S}$ is not a vertex. Then there are points $s', s'' \in \mathcal{S}$ such that s lies on the line segment from s' to s'' and $T(s)$ is a proper subset of each of $T(s')$ and $T(s'')$.*

Proof. By the definition of vertices, s cannot be the unique point in $\mathbb{R}^n$ at which the inequalities in $T(s)$ hold with equality. Therefore there is a point $\tilde{s} \in \mathbb{R}^n$ such that $\tilde{s} \neq s$ and $T(s) \subset T(\tilde{s})$. For $q = 1, \ldots, n+m$, define

$$W_q = \{t \in \mathbb{R} : h_q(s + t(\tilde{s} - s)) \geq d_q\}.$$

Notice that $0 \in W_q$ for all q, because all of the inequalities hold at s.

Since

$$h_q(s + t(\tilde{s} - s)) = h_q(s) + th_q(\tilde{s} - s) \quad \text{and} \quad h(s) \geq d_q$$

we see that either $h_q(\tilde{s} - s) = 0$ and $W_q = \mathbb{R}$ or $h_q(\tilde{s} - s) \neq 0$ and W_q is a closed, half infinite closed interval of the form $[r_q, \infty)$ of $(-\infty, r_q]$ for some real number r_q.

If $T_q \notin T(s)$, then the inequality $h_q(s) \geq d_q$ holds with strict inequality, i.e.

$$h_q(s) > d_q.$$

In this case, we will have

$$h_q(s + t(\tilde{s} - s)) = h_q(s) + th_q(\tilde{s} - s) \geq d_q$$

for all t is a small open interval which contains 0.

If $T_q \in T(s)$, then the inequalities $h_q(s) = d_q$ and $h_q(\tilde{s}) = d_q$ hold because $T_q \in T(s) \subset T(\tilde{s})$. So

$$h_q(\tilde{s} - s) = h_q(\tilde{s}) - h_q(s) = d_q - d_q = 0$$

and we conclude that in fact $W_q = \mathbb{R}$.

Therefore

$$W = \cap_{q=1}^{n+m} W_q$$

contains a small open neighborhood of 0 and is either all of $\mathbb{R}$, a half infinite closed interval or a bounded closed interval of the form $[a, b]$.

Let us show that the objective function

$$f(s) = c_1 s_1 + \cdots c_n s_n$$

is constant on W. By the definition of W, the point $s + t(\tilde{s} - s)$ satisfies all of the constraints of the linear programming problem if $t \in W$. We have

$$f(s + t(\tilde{s} - s)) = f(s) + t(f(\tilde{s}) - f(s)).$$

If $f(\tilde{s}) \neq f(s)$, then we can choose t inside a small open neighborhood of 0 contained in W to make

$$f(s + t(\tilde{s} - s)) = f(s) + t(f(\tilde{s}) - f(s)) < f(s).$$

However, this contradicts the assumption that s is a solution of the linear programming problem, since $f(s)$ must be minimal among all points which satisfy the linear equalities specified by the problem. So in fact $f(\tilde{s}) = f(s)$ and $f(s + t(\tilde{s} - s)) = f(s)$ for all t. This means in particular that f is constant on W.

We now claim that W is in fact a bounded interval of the form $[a, b]$ with $a < 0 < b$. If not, we have shown that W must contain a half infinite closed interval, so there are values of t in W which are either arbitrarily positive or arbitrarily negative. Now

$$0 = f(s - \tilde{s}) = c_1(s_1 - \tilde{s}_1) + \cdots c_n(s_n - \tilde{s}_n)$$

where all of the c_j are positive This means that some of the $s_i - \tilde{s}_i$ must be positive and some must be negative. But then once t is sufficiently large in absolute value, some entry of the vector

$$s + t(\tilde{s} - s)) = (s_1 + t(\tilde{s}_1 - s_1), \cdots, s_n + t(\tilde{s}_n - s_n))$$

would have to be negative. However, if t is in W, all the constraints of the linear programming problem are satisfied at the point $s + t(\tilde{s} - s)$, and the first n of these constraints are that the entries of $s + t(\tilde{s} - s)$ must be non-negative. Hence it is not possible that W contains t of arbitrarily large absolute value, so W cannot contains a half infinite closed interval. It must therefore be of the form $[a, b]$ with $a < b$ because it contains an open neighborhood of 0.

We now define

$$s' = s + a(\tilde{s} - s) \quad \text{and} \quad s'' = s + b(\tilde{s} - s)$$

Clearly s' and s'' satisfy all the inequalities of the linear programming problem because $a, b \in W$. We have $f(s') = f(s'') = f(s)$, so $s', s'' \in \mathcal{S}$. The line segment from s' to s'' contains s. We have shown that if $T_q \in T(s)$, then $W_q = \mathbb{R}$, so T_q holds with equality at both s' and s''. Thus $T(s) \subset T(s')$ and $T(s) \subset T(s'')$. If $T(s) = T(s')$, then each inequality T_q which is not in $T(s)$ holds with strict inequality at s'. This would mean

$$h_q(s + a(\tilde{s} - s)) > d_q.$$

Then the same is true if we replace a by any a' sufficiently close to a. This would mean that a small open neighborhood of a is contained in W_q if $T_q \notin T(s)$, and clearly such a neighborhood is contained in $W_q = \mathbb{R}$ if $T_q \in T(s)$. So we find $W = \cap_{q=1}^{n+m} W_q$ contains a small open neighborhood of a, which is impossible because W is the closed interval $[a, b]$. This shows $T(s')$ must be strictly larger than $T(s)$, and similarly $T(s'')$ must be strictly larger than $T(s)$. □

Corollary 4.5. *If $s \in \mathcal{S}$ define $m(s)$ to be the maximum of the numbers $\#T(s') - \#T(s)$ as s' ranges over the elements of $\mathcal{S}$ for which $T(s) \subset T(s')$. Then s is in $\mathcal{O}$ if and only if $m(s) = 0$.*

Proof. If s is in $\mathcal{O}$, then it is a vertex, so it must be the unique point of $\mathbb{R}^n$ at which the equalities in $T(s)$ hold with equality. Hence if $s' \in \mathbb{R}^n$ and $T(s) \subset T(s')$ then $s' = s$. This shows $T(s) = T(s')$, so $m(s) = 0$. Conversely, suppose $s \in \mathcal{S}$ and $m(s) = 0$. If s is not a vertex, then Lemma 4.4 shows $m(s) > 0$, a contradiction. So s must be a vertex, and since it is in $\mathcal{S}$, it minimizes $f(s)$. Therefore s must be in the set $\mathcal{O}$ of optimal vertices. □

Lemma 4.6. *Let $\mathcal{S}(z)$ be the the set of all linear combinations*

$$r_1 s(1) + \ldots + r_\ell s(\ell)$$

of elements $s(1), \ldots, s(\ell)$ of $\mathcal{S}$ for which $m(s(j)) \leq z$ and $r_j \geq 0$ for all j and $r_1 + \cdots + r_\ell = 1$. Then $\mathcal{S}(0) = \mathrm{Span}(\mathcal{O})$ and $\mathcal{S} = \mathcal{S}(n + m)$.

Proof. By Corollary 4.5, the elements $s \in \mathcal{S}$ for which $m(s) = 0$ are just the elements of $\mathcal{O}$, so $\mathcal{S}(0) = \mathrm{Span}(\mathcal{O})$. We have $m(s) \leq n + m$ for all $s \in \mathcal{S}$, so $\mathcal{S}(n + m) = \mathcal{S}$. □

Lemma 4.7. *For all $j \geq 0$ one has $\mathcal{S}(j) = \mathcal{S}(j + 1)$.*

Proof. Since $m(s) \leq j$ implies $m(s) \leq j+1$ we certainly have $\mathcal{S}(j) \subset \mathcal{S}(j+1)$. To prove $\mathcal{S}(j+1) \subset \mathcal{S}(j)$ it will be enough to show that every $s \in \mathcal{S}$ for which $m(s) \leq j+1$ is on a line segment between two points $s', s'' \in \mathcal{S}$ with $m(s') \leq j$ and $m(s'') \leq j$. If s is a vertex, then it is in $\mathcal{O}$ and $m(s) = 0$ so we can take $s' = s'' = s$. Otherwise, Lemma 4.4 shows that s lies on a line segment between $s', s'' \in \mathcal{S}$ with $T(s)$ a proper subset of both $T(s')$ and $T(s'')$. If $m(s') \geq j+1$ then there has to be a $s''' \in \mathcal{S}$ with $T(s') \subset T(s''')$ and $\#T(s''') - \#T(s') \geq j+1$. But then $\#T(s''') - \#T(s) \geq j+2$, contradicting $m(s) \leq j+1$. So $m(s') \leq j$ and similarly $m(s'') \leq j$, so we are done. □

End of the proof of Theorem 4.3

We showed in §3 that there is at least one solution to the linear programming problem, so $\mathcal{S}$ is not empty. By Lemmas 4.7 and 4.6 we have

$$\mathrm{Span}(\mathcal{O}) = \mathcal{S}(0) = \mathcal{S}(1) = \cdots = \mathcal{S}(n+m) = \mathcal{S}$$

which completes the proof.