MATH 210, PROBLEM SET 5

DUE IN LECTURE ON THURSDAY, APRIL 12

1. A political stability model

Suppose that $L = L(t)$, $C = C(t)$ and $U = U(t)$ represent the number of liberals, conservatives and uncommitted voters at time t. We will suppose that interactions between liberals and conservatives occur at a rate proportional to the product of their populations. Each such interaction leads to the voters becoming uncommitted with certain probabilities. Uncommitted voters spontaneously become liberal or conservative at a certain rates.

Problem 1. Suppose that there are positive constants α, β, τ and γ such that L, C and U satisfy the following differential equations:

$$\frac{dL}{dt} = -\alpha LC + \tau U \tag{1.1}$$

$$\frac{dC}{dt} = -\beta LC + \gamma U \tag{1.2}$$

$$\frac{dU}{dt} = (\alpha + \beta)LC - (\tau + \gamma)U \tag{1.3}$$

Explain why these equations correspond to the above verbal description of the evolution of L, C and U.

Problem 2. Viewing this as an autonomous system of O.D.E.'s in the vector variable

$$x(t) = \begin{pmatrix} \text{L(t)} \\ \text{C(t)} \\ \text{U(t)} \end{pmatrix}.$$

Suppose that $\alpha\gamma \neq \tau\beta$. Find all inital values $x(0)$ which are equilibria for this system.

Problem 3. Is the system ever linearly stable at the equilibrium points you found in Problem 2?

Problem 4. Show that $L(t) + C(t) + U(t)$ equals some constant κ independent of t. Suppose $\kappa > 0$ and $\alpha\gamma \neq \tau\beta$. Rewrite the system of differential equations as a system just involving L and C, using that $U = \kappa - L - C$. When you do this, which of the equilibria you bound in Problem 2 are stable for the two variable system involving only L and C? Your answer should depend on the α, β, τ and γ. Can you explain why this answer makes heuristic sense?

Remark Because there is a conserved quantity $L + C + U = \kappa$ which does not change with time, stability of the three variable system is not a natural condition. This is because such stability requires that all small variations of (L, C, U) from an equilibrium return toward the equilibrium, and most of these variations will not conserve $L + C + U$.

Extra Credit What happens in Problems 2 and 4 if $\alpha\gamma = \tau\beta$? You can turn this in at any time during the semester.