# MATH 210, PROBLEM SET 2

DUE IN LECTURE ON TUESDAY, FEBRUARY 13.

## 1. **Distances between payoff matrices.**

Let $A = (a_{i,j})_{1\le i\le n, 1\le j\le m}$ be the payoff matrix to the first player in a zero sum two person game of size $n \times m$. A different first player might have chosen a different payoff matrix $B = (b_{i,j})_{1\le i\le n, 1\le j\le m}$. This set of problems has to do with comparing the preferences reflected by the two payoff matrices $A$ and $B$.

We will measure the difference between $A$ and $B$ by treating their entries as vectors having $nm$ entries. Then we have the usual Euclidean distance function

$$|A - B| = \sqrt{\sum_{i,j} (a_{i,j} - b_{i,j})^2}.$$

**Problems:**

1. Define the dot product of $A$ and $B$ by
$$A \cdot B = \sum_{i,j} a_{i,j} \cdot b_{i,j}.$$
We add and subtract matrices entry wise, so that $A - B = (a_{i,j} - b_{i,j})$. Show that
$$|A - B|^2 = (A - B) \cdot (A - B) = A \cdot A - 2(B \cdot A) + B \cdot B = |A|^2 - 2(B \cdot A) + |B|^2.$$
2. One says $A$ and $B$ are orthogonal if $A \cdot B = 0$. Show that this is the case if and only if
$$|A - B| = |A - (-B)|.$$
Explain why this is the same as saying that the payoff matrix for the first player in the game represented by $A$ is at the same distance from the payoff matrices of each player in the game represented by $B$. (Hint: Remember that each game is zero sum!).
3. We will say that a collection $\{A_1, \ldots, A_{nm}\}$ of payoff matrices of size $n \times m$ is a Hadamard basis if for all $i$, each entry in $A_i$ is either 1 or $-1$, and if $A_i \cdot A_j = 0$ if $i \neq j$. For each $i$, let $v_i$ be the vector of length $nm$ whose components are the entries of $A_i$ taken row by row. Thus the first $m$ entries of $v_i$ are the first row of $A_i$, the second $m$ entries of $v_i$ are the second row of $A_i$ and so on. Show that $\{A_1, \ldots, A_{nm}\}$ is a Hadamard basis for payoff matrices of size $n \times m$ if and only if the $nm \times nm$ matrix $V$ whose $i$-th row is $v_i$ is a Hadamard matrix, in the sense that every entry in $V$ is either $\pm 1$ and distinct rows of $V$ have dot product 0.

4. Show that when $n = m = 2$, the matrices

$$A_1 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \quad A_2 = \begin{array}{|c|c|} \hline 1 & -1 \\ \hline -1 & 1 \\ \hline \end{array} \quad A_3 = \begin{array}{|c|c|} \hline 1 & -1 \\ \hline 1 & -1 \\ \hline \end{array} \quad A_4 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline -1 & -1 \\ \hline \end{array}$$

form a Hadamard basis. Write down the $4 \times 4$ Hadamard matrix $V$ corresponding via problem # 3 to this basis. Notice that $A_2$, $A_3$ and $A_4$ were the components of the payoff in the truth versus lying game discussed in class on Jan. 23.

5. Suppose $\{A_1, \ldots, A_{nm}\}$ is a Hadamard basis for the payoff matrices of size $nm$. A standard result in linear algebra is that the only $n \times m$ matrix $C$ for which $C \cdot A_i = 0$ for all $i$ is the matrix which has all entries equal to 0. We will take this fact for granted.[1] Use this fact to show that if $B$ is any payoff matrix of size $nm$, then

$$B = \sum_{k=1}^{nm} c_k A_k \quad \text{with} \quad c_k = \frac{B \cdot A_k}{nm}.$$

Explain why this says that the preferences of player 1 in the game can be broken down into a linear combination of the preferences represented by the $A_i$.

6. In the $n = m = 2$ case, describe a two-person two-option game between yourself and someone else, and write down a payoff matrix that would correspond to this. Then use problem #5 to break your payoff matrix down into a sum of the matrices in problem # 4. Finally, what does this say about how your payoff preferences break down into a sum of:
   i. A happy or sad benefit that comes along no matter what,
   ii. Wanting or not wanting to pick the same option number as the other player,
   iii. Taking only the option choice of the other player into account, and
   iv. Taking only your option choice into account.

**Comments:** By problem # 3, finding Hadamard bases for games of size $n \times m$ is the same as finding Hadamard matrices $V$ of size $nm \times nm$. Two Hadamard matrices $V$ and $V'$ are said to be equivalent if one results from the other by switching the orders of rows or columns after multiplying some rows and columns by $\pm 1$. It is known that when $n = m = 2$, all Hadamard matrices are equivalent in this sense, but this will not always be the case for larger $nm$. It is a major open research problem to prove that for all integers $nm$ divisible by 4, there is a Hadamard matrix $V$ of size $nm \times nm$ (!).

## 2. **A three option two person zero sum game.**

The rock-paper-scissors game has two players who simultaneously choose between three options; rock, paper or scissors. If they choose the same option, the outcome is a draw, resulting in a payoff of 0 to both players. Otherwise, the payoff to each player is either 1 or $-1$ according to the following rule: rock beats scissors, scissors beats paper and paper beats rock.

[1] If you know some linear algebra, try proving this for extra credit.

**Problems:**

**1.** Write down the payoff matrix for the first player (player I) of this game. There should be three rows and three columns corresponding to the choices of the two players.

**2.** Find the maximin and minimax of this payoff matrix from the point of view of the first player. Is there a saddle point?

**3.** Suppose that the two players pursue mixed strategies represented by the vector $P = (p_1, p_2, p_3)$ for player I and the vector $Q = (q_1, q_2, q_3)$ for player II, where

$$0 \leq p_i \leq 1 \quad \text{and} \quad 0 \leq q_j \leq 1$$

for all $i$ and $j$ and

$$p_1 + p_2 + p_3 = 1 = q_1 + q_2 + q_3.$$

**3a.** Suppose that player I picks $(p_1, p_2, p_3) = (1/3, 1/3, 1/3)$. Show that for all choices of $Q = (q_1, q_2, q_3)$ by player II, the expected payoff of the game to player I is 0. Recall that the expected payoff is the sum over all possible pairs of choices of the probability of that choice times the payoff associated to this pair. You may find it useful when computing this expected payoff to group together the terms which correspond to a fixed choice by player II.

**3b.** Suppose now that player I picks a vector $P = (p_1, p_2, p_3)$ as above which is different from $(1/3, 1/3, 1/3)$. How could player II pick $Q = (q_1, q_2, q_3)$ to take advantage of $P \neq (1/3, 1/3, 1/3)$ in order to make the payoff to player I less than 0? Can player II do this by choosing a pure strategy, i.e. one for which $q_j = 1$ for some $j$?

**3c.** Let $E(P, Q)$ be the expected value of the payoff to player I from if player I chooses $P$ and player II chooses $Q$ as above. Conclude from problems (3a) and (3b) that

$$v_I = \max_P(\min_Q\ E(P, Q)) = 0$$

where $P$ and $Q$ ranges over all allowable choices.

**3d.** Using similar ideas, show that

$$v_{II} = \min_Q(\max_P\ E(P, Q)) = 0$$

and that this value is achieved when player II picks $Q = (1/3, 1/3, 1/3)$. Conclude that the value of the game is 0 and that the best mixed strategies for each player are to choose each option a third of the time.