MATH 210, PROBLEM SET 9

DUE IN LECTURE ON WEDNESDAY, NOV. 17

1. **The central limit theorem**

In class we talked about the central limit theorem for $n \geq 1$ independent random variables $X_1, \ldots, X_n$ having the same distribution function. Let $\mu = E(X_i)$ and $\sigma = \sigma(X_i) = \sqrt{\mathrm{Var}(X_i)}$ be the mean and standard deviation of each of the X_i. Define

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}.$$

The central limit theorem, with Berry's error term, says

$$|\Pr(Y_n \leq x) - \int_{-\infty}^{x} \phi(u)du| \leq \frac{3\rho}{\sigma^3\sqrt{n}} \tag{1.1}$$

when $\rho = E(|X_i - \mu|^3)$, where $\phi(u) = \frac{1}{\sqrt{2\pi}}e^{-u^2/2}$ is the probability density function of the normal distribution.

1. The newspaper columnist Marilyn Vos Savant often claimed that she had an I. Q. of 220, which would be 8 standard deviations above the mean. Suppose that the I.Q. test supporting this had n questions. Suppose that scores of people taking the test on questions 1 through n are represented by independent random variables $X_1, \ldots, X_n$, with $\mathrm{Prob}(X_i = 1) = 1/2$ being the probability of a correct answer and $\mathrm{Prob}(X_i = 0) = 1/2$ being the probability of a wrong answer. What is the smallest number n of questions on the exam such that Marilyn Vos Savant's total score $X = X_1 + \cdots + X_n$ could have been 8 standard deviations above the mean? (Hint: You don't need to use the central limit theorem for this, just the formula for $\sigma(X)$.)

2. When n is your answer to question 1, what is the probability that a person has the same score on the exam as Marilyn Vos Savant? If the population of the earth is $6 \cdot 10^9$, what is the probability of finding a person with this high a score on earth, given that the exam could be modeled as in problem # 1? How would you explain your conclusions?

3. Suppose we now want to use the central limit theorem to approximate the probability found in question #2. Let $x = 8$ in (1.1). How large should we make n in order for to make the error term on the right hand side of (1.1) less than the probability you found in question #2 of a person having a score equal to Marilyn Vos Savant's? (This n will be considerably larger than the answer you find in question #1.)

   Comment: This is another illustration of how the known error estimates for the central limit theorem do not work well at large distances from the mean unless n is exceedingly large.

2. **Stirling's formula**

One form of Stirlings formula is that if $n \geq 1$ is an integer, then

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\lambda_n} \tag{2.2}$$

where

$$\frac{1}{12n+1} < \lambda_n < \frac{1}{12n} \tag{2.3}$$

4. Use Stirling's formula to show that the probability of getting exactly $n/2$ heads on flipping a fair coin an even number n of times is equal to

$$\sqrt{\frac{2}{n\pi}}\, e^{s_n} \tag{2.4}$$

where

$$\frac{1}{12n+1} - \frac{1}{3n} < s_n < \frac{1}{12n} - \frac{2}{6n+1}.$$

(This corrects a formula we talked about in class.)