MATH 210, PROBLEM SET 8

DUE IN LECTURE ON WEDNESDAY, NOV. 10

1. Independent random variables

Let $X : S \to \mathbb{R}$ be a random variable. The distribution function of X is the function $F_X : \mathbb{R} \to \mathbb{R}$ defined by

$$F_X(t) = \text{Prob}(X \leq t).$$

Random variables $X_1, \ldots, X_n$ are independent if for all n-tuples of closed intervals $[a_1, b_1], \ldots, [a_n, b_n]$, one has

$$\text{Prob}(X_i \text{ is in } [a_i, b_i] \text{ for all } i = 1, \ldots, n) = \prod_{i=1}^{n} \text{Prob}(X_i \in [a_i, b_i]).$$

1. Suppose that at time $t = 0$ you begin watching a TV program in which a liberal, a moderate and a conservative are discussing politics. Let X_1 be the amount of time before the liberal says the word "extremists." Let X_2 be the amount of time before the moderate stands up and walks out. Let X_3 be the amount of time before the conservative says "founding fathers." You decide you are going to switch off the program as soon as either:
   (i) The moderate walks out, or
   (ii) The liberal has said "extremist" and the conservative has said "founding fathers".

   Let X be the random variable giving the time you switch off the program. Suppose that X_1, X_2 and X_3 are independent. How can you write down the distribution function F_X of X in terms of the distribution functions F_{X_1}, F_{X_2} and F_{X_3}?

   (Hint: To find $F_X(t) = \text{Prob}(X \leq t)$, try to write down the event that $X \leq t$ in terms of conditions satisfied by X_1, X_2 and X_3. Then try to write this event as a disjoint union of events to which you can apply the independence of X_1, X_2 and X_3.)

2. Write down some examples form your own experience of independent random variables, and of random variables which are not independent. As challenge, try to think of a real life situation in which there are three random variables which are pairwise independent but which are not independent.

2. Expectations

Suppose that a movie producer is considering whether to show their new film, "Saw LXIII: Stubbed toes and Paper cuts," in theaters. The producer tries to determine whether this will be profitable for them in the following way.

Suppose N is the total number of people who will see the movie at some point. Let X be the random variable given by the average time in weeks that someone sees the movie, assuming that they see it at some point. Assume that the probability density function of X has the form

$$f_X(t) = c_1 e^{c_2 t}$$

for some constants c_1 and c_2. Thus the probability that someone who will eventually see it will go to see it between times a and b is

$$\mathrm{Prob}(a \leq X \leq b) = \int_a^b f_X(t)dt.$$

when $0 \leq a \leq b$.

3. Suppose we measure time in units of weeks, and that the odds that someone sees the move in week two (from time 1 to time 2) are half what they are in week one (from time 0 to time 1) if they see the movie at all. What should c_1 and c_2 be in the formula for $f_X(t)$?
4. Using c_1, c_2 and N write down a formula for the expected number $h(T)$ of people who will have seen the move by time T.
5. Suppose that the cost of showing the movie for T weeks is given by $g(T) = I + cT$ dollars, where I is the start up cost and c is a constant. Suppose each person who sees the move brings in r dollars of revenue. How would you use the functions $h(T)$, $g(T)$ and r to determine whether there is a number T of weeks to show the movie which will be profitable for the movie maker? (Try to push the calculus as far as you can to arrive at a readily testable formula.) Is it ever profitable for the theater to show the movie forever?

3. The semester project

6. Describe an idea you'd like to pursue for your semester project. Anything of interest to you which has some mathematical content is worth considering, and I'll be happy to talk with you and your group about this. Some ideas which have come up during this semester are:
   i. Follow through on optimizing the expected rank of the person selected by the interview procedure we talked about in class.
   ii. Look into the statement made on page 81 of the book "How math can save your life" about the effect on life expectancy of freeway driving.
   iii. How can probability and game theory be used to come up with a system for settling legal cases?

   But please don't feel constrained to follow these ideas!