MATH 210, PROBLEM SET 10

DUE IN LECTURE ON WEDNESDAY, NOV. 24

1. **Poisson distributions**

In class we discussed Poisson processes. These are systems in which events occur at some average rate λ per unit time and the number of events in a given time interval is independent of the number of events in any other disjoint time interval. The random variable X_T giving the number of events which will occur between time 0 and time T has probability density function

$$f_{X_T}(j) = \frac{e^{-\lambda T}(\lambda T)^j}{j!} \quad \text{for} \quad 0 \le j \in \mathbb{Z} \tag{1.1}$$

This was derived by chopping $[0, T]$ into a large number n of subintervals and taking a limit of the appropriate Bernoulli random variables associated to independent coin flips over the subintervals. The random variable Y giving the expected wait time till the first event satisfies:

$$F_Y(T) = \mathrm{Prob}(Y \le T) = 1 - \mathrm{Prob}(X_T = 0) = 1 - f_{X_T}(0) = 1 - e^{-\lambda T}.$$

This Y is said to have an exponential probability distribution with parameter λ. The probability density function of Y is

$$f_Y(T) = \frac{d}{dT}F_Y(T) = \frac{d}{dT}(1 - e^{-\lambda T}) = \lambda e^{-\lambda T}.$$

1. For each of the following phenomena, discuss the arguments for and against modeling the phenomena as a Poisson process when T ranges between 0 and 10 years. If you decide a Poisson process is reasonable, what should λ be, and what is the probability that there will be no events during the first day of observations?

   a. Members of the U.S. population of $3 \cdot 10^8$ people come down with a certain exotic illness at the rate of 100 people per year, which they will continue to have. Suppose we say an event consists of one person in the U.S. coming down with the illness. You can neglect the fact that the overall population changes over time.

   b. After a space probe returns from Mars, members of the U. S. population of $3 \cdot 10^8$ people begin to become zombies at the rate of 10^8 people per year. Once they do so, they stay zombies. Suppose we say that an event is that a member of the initial population of humans becomes a zombie.

   c. Same as problem [b], but due to a vaccine, people who become zombies are cured and return to human status almost immediately.

2. **Continued fractions**

2. Suppose
$$r = \frac{\ln(3/2)}{\ln(2)} = 0.584962500721156$$
Wolfram alpha produces the beginning of the continued fraction for r:
$$r = [0; 1, 1, 2, 2, 3, \ldots]$$
Show using the above decimal expansion for r and the algorithm discussed in class why the continued fraction begins in this way. Then find the convergents p_i/q_i to the continued fraction for $0 \leq i \leq 5$. (This will be relevant to one of the course projects.)

3. Prove that the continued fraction for $\sqrt{2}$ equals $[1; 2, 2, 2, 2, \ldots]$, where after the initial 1 one has all 2's.