**MATH 210, EXTRA CREDIT PROBLEMS # 1**

DUE AT ANY TIME DURING THE SEMESTER

## 1. Two-person two-option zero sum games.

The object of these exercises is to work through some details of the analysis of two-person two-option zero sum games discussed in class. Each subsection recalls some of the theory and notation discussed in class. Homework problems are highlighted between vertical lines.

1.1. **Dominant strategies.** Recall that payoff matrix for Player I has the form:

| | Player II option 1 | Player II option 2 |
|---|---|---|
| Player I option 1 | $a_{1,1}$ | $a_{1,2}$ |
| Player I option 2 | $a_{2,1}$ | $a_{2,2}$ |

The payoffs to Player II are the negatives of the entries in this matrix.

Player I has a dominant strategy if each entry in one of the rows of the matrix is $\geq$ the corresponding entry in the other row.

Player II has a dominant strategy if and only if each entry in one column of the matrix is $\leq$ the entry in the other column.

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**Problem:**

1. Show neither player has a dominant strategy if and only if (i) $a_{1,1}-a_{1,2}$ and $a_{2,1}-a_{2,2}$ have opposite signs, and (ii) $a_{1,1} - a_{2,1}$ and $a_{1,2} - a_{2,2}$ have opposite signs.

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1.2. **Payoffs for mixed strategies.** We suppose that Player I chooses option #1 with probability $p$ and option #2 with probability $1 - p$. Player II chooses option # 1 with probability $q$ and option # 2 with probability $1 - q$. Since they make these choices independently of one another, the probability of various combinations of choices is given by the following matrix:

| | Player II option 1 prob. $q$ | Player II option 2 prob. $1 - q$ |
|---|---|---|
| Player I option 1 prob. $p$ | $pq$ | $p(1 - q)$ |
| Player I option 2 prob. $1 - p$ | $(1 - p)q$ | $(1 - p)(1 - q)$ |

The expected payoff to player I is then the sum over the various possible combinations of choices of the product of the probability of that combination times the payoff of that combination. This works out to

$$\begin{aligned} E(p,q) &= pqa_{1,1} + p(1-q)a_{1,2} + (1-p)qa_{2,1} + (1-p)(1-q)a_{2,2} \\ &= (a_{1,1} - a_{1,2} - a_{2,1} + a_{2,2})pq + (a_{1,2} - a_{2,2})p + (a_{2,1} - a_{2,2})q + a_{2,2} \\ &= \Delta pq - np - mq + r \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} \Delta &= a_{1,1} - a_{1,2} - a_{2,1} + a_{2,2} \\ n &= a_{2,2} - a_{1,2} \\ m &= a_{2,2} - a_{2,1} \\ r &= a_{2,2} \end{aligned} \tag{1.2}$$

Player I wishes to find

$$v_I = \max_{0 \le p \le 1} \left( \min_{0 \le q \le 1} E(p,q) \right) \tag{1.3}$$

since this represents the best expected return they can achieve against any strategy of player $II$. Specifically, if they choose a value $p_0$ such that

$$v_I = \min_{0 \le q \le 1} E(p_0,q)$$

then player I is guaranteed an expected return of at least $v_I$ against any choice of $q$ by player $II$. If $p$ is any other choice which Player I might make, then

$$v_I \ge \min_{0 \le q \le 1} E(p,q)$$

so Player II can pick a $q$ which will prevent the payoff $E(p,q)$ from being larger than $v_I$.

Similarly, Player II wishes to find

$$v_{II} = \min_{0 \le q \le 1} \left( \max_{0 \le p \le 1} E(p,q) \right) \tag{1.4}$$

since this represents the minimal expected payoff that they can hold Player I to in the game.

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**Problems**

**2.** Suppose that Player I has dominant stategy given by option 1, so that $a_{1,1} \ge a_{2,1}$ and $a_{1,2} \ge a_{2,2}$. Show that for all $p, q$ such that $0 \le p \le 1$ and $0 \le q \le 1$ one has

$$E(1,q) = qa_{1,1} + (1-q)a_{1,2}$$

$$\ge q(pa_{1,1} + (1-p)a_{2,1}) + (1-q)(pa_{1,2} + (1-p)a_{2,2}) \tag{1.5}$$

$$= E(p,q). \tag{1.6}$$

Deduce from this that

$$\begin{aligned} v_I &= \max_{0 \le p \le 1} \left( \min_{0 \le q \le 1} E(p,q) \right) \\ &= \min_{0 \le q \le 1} E(1,q) \\ &= \min_{0 \le q \le 1} \left( qa_{1,1} + (1-q)a_{1,2} \right) \\ &= \min(a_{1,1}, a_{1,2}) \end{aligned} \tag{1.7}$$

Then show that

$$v_{II} = \min_{0\le q\le 1}(\max_{0\le p\le 1} E(p,q)) = \min_{0\le q\le 1} E(1,q) = v_I \tag{1.8}$$

This shows that $v_I = v_{II}$ if Player 1 has dominant option # 1, and that this is payoff is achieved when Player I chooses option 1 and Player II chooses the best pure strategy against this, corresponding to setting $q = 1$ if $a_{1,1} \le a_{1,2}$ and $q = 0$ if $a_{1,2} < a_{1,1}$. The proof that $v_I = v_{II}$ results from a choice of pure strategies in the other cases in which one player has a dominant strategy is similar, so we'll not repeat the arguments.

**3.** Suppose that

$$\Delta = a_{1,1} - a_{1,2} - a_{2,1} + a_{2,2} = 0$$

in (1.2). Deduce from this that

$$a_{1,1} - a_{1,2} = a_{2,1} - a_{2,2}$$

and

$$a_{1,1} - a_{2,1} = a_{1,2} - a_{2,2}.$$

Explain why this means that both players have a dominant strategy. Then you can conclude that $v_I = v_{II}$ is achieved by pure strategies using problem #2.

1.3. **Completion of the proof of the minimax theorem, namely: $v_I = v_{II}$.** So far we have shown that $v_I = v_{II}$ if one of the two players has a dominant strategy (problem 2), and that if neither player has a dominant strategy then $\Delta \ne 0$ (problem 3). We suppose now that neither player has a dominant strategy. In class we proved the identity

$$E(p,q) = \Delta pq - np - mq + r = \Delta(p - \frac{n}{\Delta})(q - \frac{m}{\Delta}) + E(\frac{n}{\Delta}, \frac{m}{\Delta}) \tag{1.9}$$

$$= \Delta st + E(\frac{n}{\Delta}, \frac{m}{\Delta}) = \Delta st + E(p_0, q_0) \tag{1.10}$$

when

$$s = p - \frac{n}{\Delta} = p - p_0 \quad \text{and} \quad t = q - \frac{m}{\Delta} = q - q_0. \tag{1.11}$$

This identity (1.9) results form checking that the coefficients of $pq$, $p$ and $q$ on both sides of the first line of (1.9) are the same and by then checking that the constant term is correct by evaluating both sides at $p = -\frac{n}{\Delta}$ and $q = -\frac{m}{\Delta}$. The second line in (1.9) is just changing variables using (1.11). The change of variables from $(p,q)$ to $(s,t)$ lead in class to:

$$v_I = \max_{0\le p\le 1}(\min_{0\le q\le 1} E(p,q)) = \max_{-p_0\le s\le 1-p_0}(\min_{-q_0\le t\le 1-q_0} \Delta st + E(p_0,q_0)) \tag{1.12}$$

and

$$v_{II} = \min_{0\le q\le 1}(\max_{0\le p\le 1} E(p,q)) = \min_{-q_0\le t\le 1-q_0}(\min_{-p_0\le s\le 1-p_0} \Delta st + E(p_0,q_0)) \tag{1.13}$$

**Problems.**

**4a.** Using the formulas for $n$ and $m$ in (1.2), show that

$$p_0 = \frac{n}{\Delta} = \frac{a_{2,2} - a_{1,2}}{a_{2,2} - a_{1,2} + a_{1,1} - a_{2,1}} \tag{1.14}$$

and

$$q_0 = \frac{m}{\Delta} = \frac{a_{2,2} - a_{2,1}}{a_{2,2} - a_{2,1} + a_{1,1} - a_{1,2}} \tag{1.15}$$

Then use problem # 1 to show that becaue we have assumed that neither player I or player II have a dominant strategy, one has $0 < p_0 < 1$ and $0 < q_0 < 1$.

**4b.** Using the inequalities on $p_0$ and $q_0$ in problem (4a), show that $(s_0, t_0) = (0, 0)$ is then an allowable value for $(s, t)$ in the formulas (1.12) and (1.13). Then show that in fact

$$v_I = v_{II} = E(p_0, q_0).$$

Hints: We know

$$v_I = \max_{-p_0 \le s \le 1-p_0} (\min_{-q_0 \le t \le 1-q_0} \ \Delta st + E(p_0, q_0)) \ge E(p_0, q_0)$$

since the maximum over $s$ is at least as large as what one gets when one lets $s = s_0 = 0$, and the latter value is $E(p_0, q_0)$. Argue that if $-p_0 \le s \le 1 - p_0$ and $s \ne 0$, then one can always choose $t$ near 0 so that $\Delta st$ is negative. Then explain why this gives $v_I = E(p_0, q_0)$. The argument for $v_{II}$ is similar.

**4c.** Suppose as above that neither player I or II have a dominant strategy. Explain why the numbers $p_0$ and $q_0$ in Problem 4a above are given by

$$p_0 = \frac{c_1}{c_1 + c_2} \quad \text{and} \quad q_0 = \frac{d_1}{d_1 + d_2}$$

when we let $c_1, c_2, d_1, d_2$ be the numbers computed by the following table discussed in class:

| | Player II option 1 | Player II option 2 | |
|---|---|---|---|
| Player I option 1 | $a_{1,1}$ | $a_{1,2}$ | $c_1 = \|a_{2,1} - a_{2,2}\|$ |
| Player I option 2 | $a_{2,1}$ | $a_{2,2}$ | $c_2 = \|a_{1,1} - a_{1,2}\|$ |
| | $d_1 = \|a_{1,2} - a_{2,2}\|$ | $d_2 = \|a_{1,1} - a_{2,1}\|$ | |