Can You Hear Me Now?

An Introduction to the Mathematics of Hearing

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Some Questions

- How does hearing work?
- What are the important structures and mechanisms of the auditory system?
- How can mathematics improve our understanding of hearing?
- What is a cochlear implant and what role can mathematical modeling play in improving implant performance?

Physical Basis of Sound

Any vibrating object produces sound. The sound we hear comes from pressure waves propagating through the air.

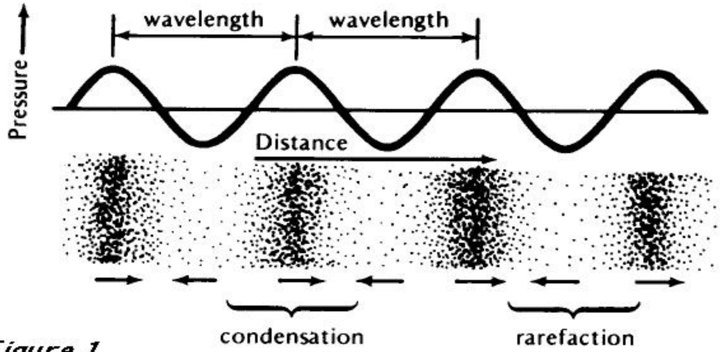
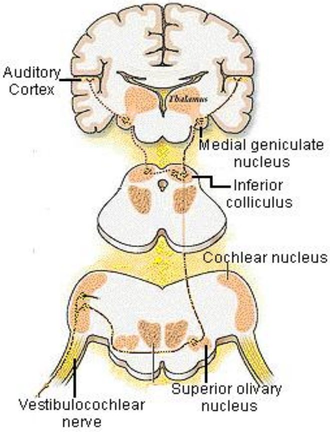


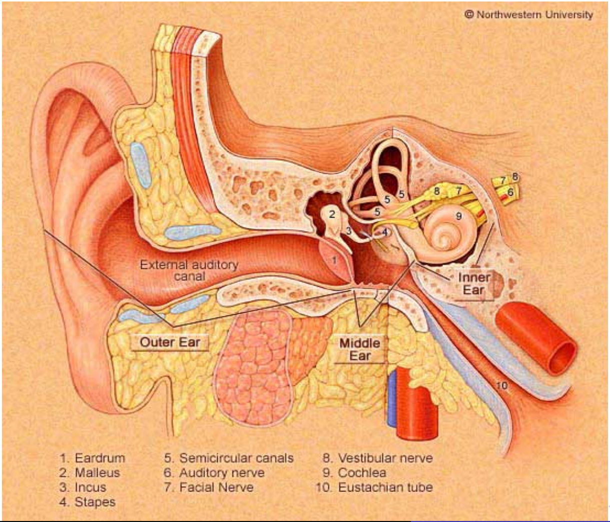
Figure 1

Perceptual Basis of Sound

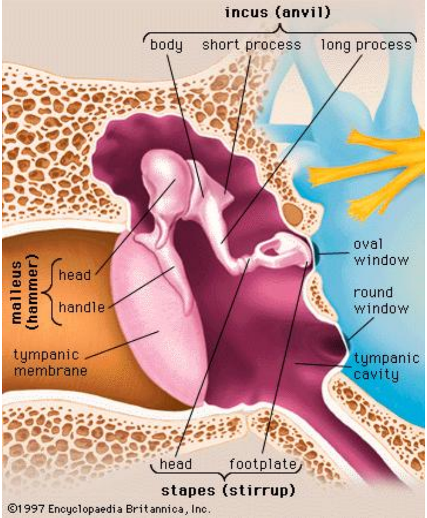
Our perception of sound is due to neural response in the auditory cortex of the brain.



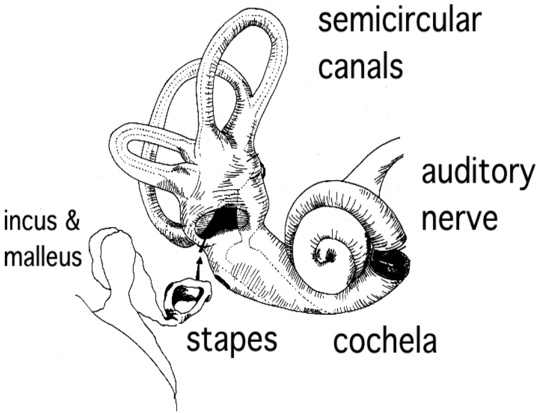
Anatomy of the Ear



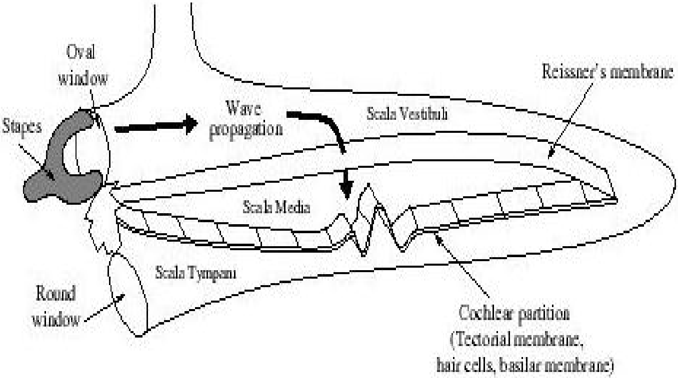
The Middle Ear



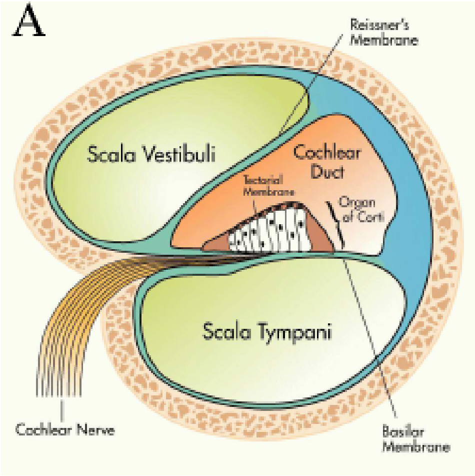
The Inner Ear



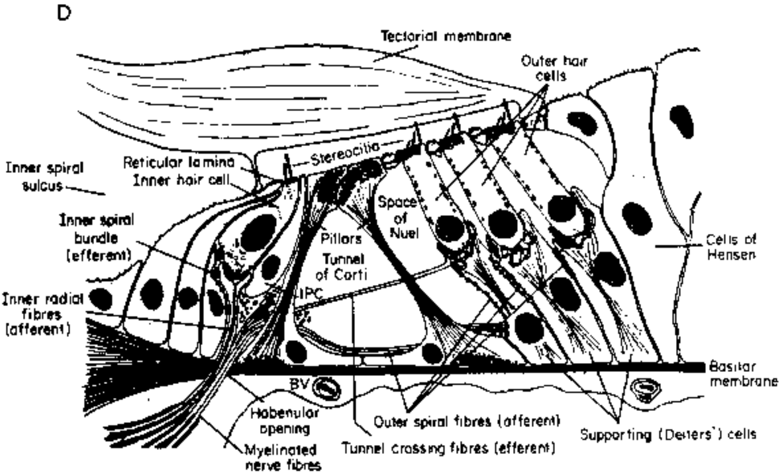
The Cochlea


Oval window
Stapes
Wave propagation
Scala Vestibuli
Reissner's membrane
Scala Media
Scala Tympani
Round window
Cochlear partition
(Tectorial membrane,
hair cells, basilar membrane)

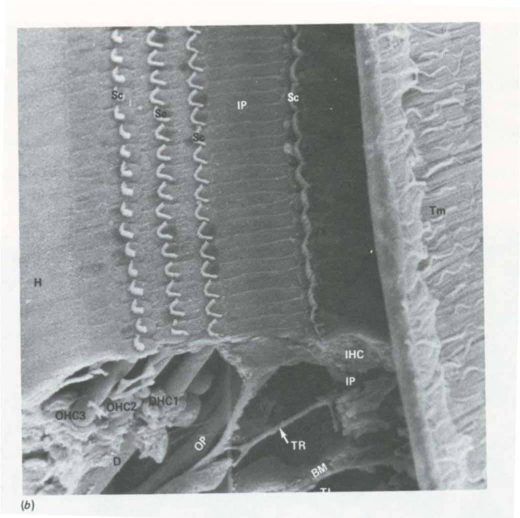
Cross-Section of the Cochlea



Detail of Cochlear Partition



Hair Cells and Transduction



Cochlear Mechanics: Place Theory of Hearing

- The 1961 Nobel Prize in Medicine was awarded to Georg von Bekesy for his lifetime of experimenatl research into cochlear mechanics.
- Two of his key findings were the *traveling wave* and *tonotopic response* of the basilar membrane.
- The location of maximum BM vibration is determined by the frequency of the stimulus.
- Modern techniques have enabled researchers to observe the response of cochleae in live subjects, leading to the discovery of the *active mechanism*.

Cochlear Mechanics: What to Model?

- Traveling Wave
- Frequency Dependence of Traveling Wave Envelope
- Spiral Geometry in Three Dimensions
- Micromechanics of the Organ of Corti
- Outer Hair Cell Motility and Active Mechanism

Cochlear Mechanics: Fluid Dynamics

Since the cochlear fluid is irrotational the velocity field is:

$$\mathbf{v} = \nabla\Phi.$$

Since the fluid is incompressible, linear, and inviscid, the fluid dynamics reduce from Navier Stokes to the potential equation:

$$\nabla^2\Phi = 0.$$

Conservation of momentum requires:

$$P_1 + \rho\frac{\partial\Phi}{\partial t} = 0$$

where P_1 is the pressure in the upper chamber, ρ is fluid density.

Cochlear Mechanics: Boundary Conditions

Consider the interface at $z = 0$ and let n be the displacement of the basilar membrane normal to the interface.
Continuity at the interface requires:

$$\frac{\partial n}{\partial t} = \frac{\partial \Phi}{\partial z}.$$

Assume the mechanics of the membrane are governed by:

$$m\frac{\partial^2 n}{\partial t^2} + \beta\frac{\partial n}{\partial t} + \kappa n = P_2 - P_1 = -2P_1.$$

This is a commonly used model that assumes the membrane vibrates as a series of uncoupled springs of mass m, damping β and stiffness κ. β and κ vary exponentially with distance from the base of the cochlea.

Cochlear Mechanics: Simplifying the BVP

We will restrict ourselves to the steady state response of the cochlea to pure tone of frequency ω. Assuming linearity of the system we can transform the BVP:

Define $\Phi = e^{i\omega t}\phi$ and $n = e^{i\omega t}\eta$, and $P_1 = e^{i\omega t}p_1$

Then the BVP for the upper chamber becomes:

$$\begin{aligned}
\nabla^2\phi &= 0 \\
\frac{\partial\phi}{\partial z} &= \frac{2\rho i\omega\phi}{Z} \text{ at } z = 0 \text{ (Basilar Membrane)} \\
\frac{\partial\phi}{\partial x} &= \omega \text{ at } x = 0 \text{ (Oval Window)} \\
\frac{\partial\phi}{\partial \hat{n}} &= 0 \text{ at rigid walls}
\end{aligned}$$

where $Z = i\omega m + \beta + \frac{\kappa}{i\omega}$.

Finite Differences

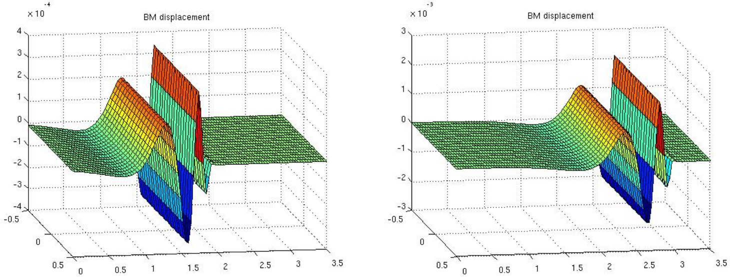
One method for numerically solving diffential equations is to use *finite difference operators*. Recall the definition of a derivative:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ if the limit exists.}$$

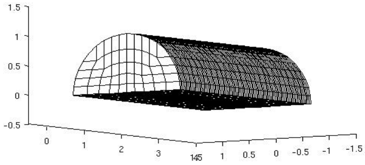
On a computer, we *approximate* derivatives by computing this fraction for small (but finite) h. When solving differential equations, this results in the need to solve systems of equations.

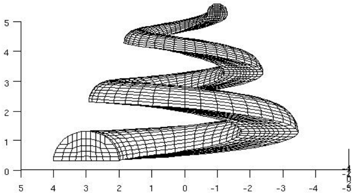
Traveling Waves in a Straight Rectangular Cochlea

Basilar Membrane Response to 1000Hz and 200Hz Stimuli



Physical Domains using Logically Rectangular Grids

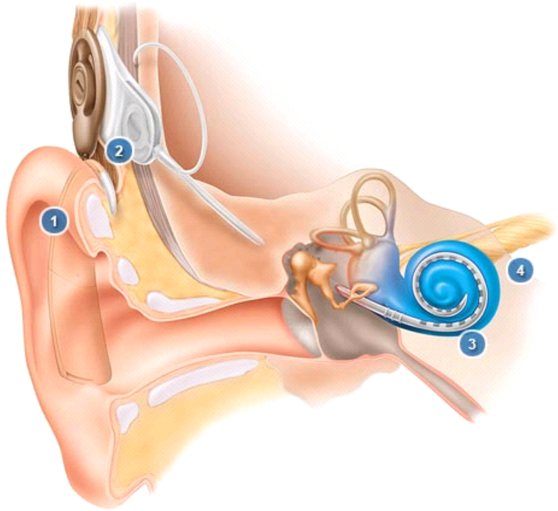




Cochlear Implants

- As we have seen, healthy hearing relies on complicated system of finely tuned structures in the ear to convert sound (pressure waves) into a neural response that is sent to the brain.
- For many individuals with hearing loss, some part of this transduction process does not work.
- Cochlear Implants work to restore hearing by stimulating auditory nerves directly with electical pulses.

Cochlear Implants



Cochlear Implants

- According to the National Institute on Deafness and Other Communication Disorders, the FDA reported in 2005 that nearly 100,000 people worldwide have received cochlear implants. In the United States, roughly 22,000 adults and nearly 15,000 children have received implants.
- Successful cochlear implants can allow otherwise deaf patients to communicate without lip-reading or other visual clues, talk on the telephone, and help deaf children develop speech and language skills.

Mathematical Model

Mathematical models of cochlear implants require two components:

- Electrical Field Model
- Neural Excitation Model

Electrical Field

Electrostatic field generated by electrodes can be modeled by Poisson's Equation with point sources representing the electrodes:

$$\nabla^2 \phi = \delta(x - x_0, y - y_0, z - z_0).$$

In simplified geometries, exact solutions are possible. To capture full biological complexity (spiral shape, characteristics of tissues, bones, etc.) numerical methods must be employed.

Neural Excitation

In response to an electric field, neurons respond by producing *action potentials* that are transmitted from neuron to neuron until they reach the brain.

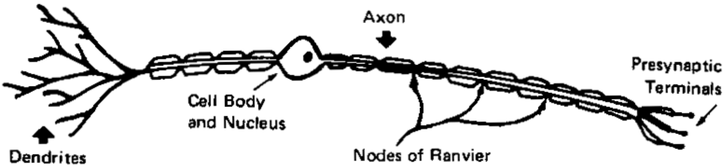


Figure 2.32 Schematic drawing of a typical bipolar sensory neuron.

Modeling Action Potentials

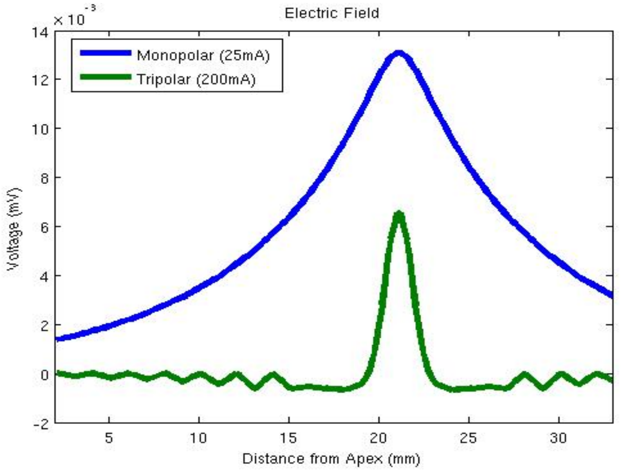
The 1963 Nobel Prize in Medicine was awarded to Hodgkin and Huxley for their research into the dynamics of action potentials. They developed the following set of Ordinary Differential Equations:

$$
\begin{aligned}
\frac{dV}{dt} &= \frac{1}{C_m}(I - (I_{Na} + I_K + I_L)) \\
\frac{dm}{dt} &= \alpha_m(1 - m) - \beta_m m \\
\frac{dh}{dt} &= \alpha_h(1 - h) - \beta_h h \\
\frac{dn}{dt} &= \alpha_n(1 - n) - \beta_n n.
\end{aligned}
$$

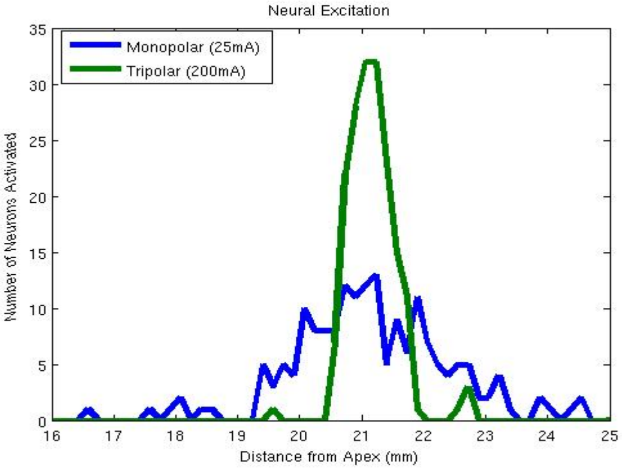
Spatial Selectivity

- Perception of frequency is related to location of stimulated neurons (tonotopy)
- Original cochlear implants were *monopolar*, thus a broad cluster of neurons is stimulated in response to sound.
- More restricted neural excitation can be achieved through *tripolar* cochlear implants.

Monopolar vs Tripolar Electric Field



Monopolar vs Tripolar Neural Excitation



Conclusion

- Our ability to hear relies on a complicated system of biological processes that converts sound waves to vibrations in the middle and inner ear which trigger neural responses that are sent to the brain for processing.
- The auditory system is not fully understood and there are numerous opportunities for mathematical modeling to contribute to our understanding of hearing.
- The full complexity of biological systems can never be captured by (tractable) mathematical equations, but well formulated models can still improve our understanding of physiological and other biological systems.