K-THEORY OF INFINITY CATEGORIES

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ABSTRACT. Notes from an expository talk given in the algebraic $K$-theory seminar at UPenn, spring 2020.

0.1. References.

- Barwick, *On the Algebraic $K$-Theory of Higher Categories*
- Blumberg, Gepner, Tabuada (BGT), *A Universal Characterization of Higher Algebraic $K$-Theory*
- Brasca, $K$-theory of Waldhausen categories
- Lurie, MATH281, Lectures 14, 16
- Lurie, *Higher Topos Theory*
- Lurie, *Kerodon*
- Joyal, *The Theory of Quasi-Categories and its Applications*

1. Infinity Categories

1.1. Intuition. Let $\textbf{Cat}$ be the category of all small categories. This had the following data:

<table>
<thead>
<tr>
<th>small categories</th>
<th>objects</th>
</tr>
</thead>
<tbody>
<tr>
<td>functors</td>
<td>morphisms</td>
</tr>
<tr>
<td>natural transformations</td>
<td>morphisms between morphisms.</td>
</tr>
</tbody>
</table>

If we consider natural transformations as part of the data of $\textbf{Cat}$, we have considerably more data than an ordinary category (often called a 1-category). Here we call $\textbf{Cat}$ a 2-category, and we can call the natural transformations 2-morphisms.

1.2. Enrichment. The reason we were able to reasonably talk about 2-morphisms in $\textbf{Cat}$ is due to the following observation:

For any $\mathcal{C}, \mathcal{D} \in \textbf{Cat}$, we have that $\text{Fun}(\mathcal{C}, \mathcal{D})$ is a 1-category, whose objects are functors and whose morphisms are natural transformations.

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This leads to the following ad hoc definition: a \(2\)-category is any category whose homs are \(1\)-categories.

To make this more explicit we would need the notion of enriched categories, which we won’t discuss here. However to soup this up to a legitimate definition, we would say a \(2\)-category is any category enriched in a category of \(1\)-categories.

1.3. Infinity categories. Inductively, we think about \(n\)-categories as being any category enriched in a category of \((n - 1)\)-categories. That means homs in an \(n\)-category are \((n - 1)\)-categories, and we think about \(n\)-morphisms as being morphisms between \((n - 1)\)-morphisms.

If we have \(n\)-morphisms for every \(n\), then we say that we have an \(\infty\)-category. We remark though that we can always view any category as an \(\infty\)-category by just letting all the higher morphisms be the identity.

An \((n, r)\)-category is a category for which all \(k\)-morphisms with \(k > n\) are trivial, and all \(k\)-morphisms with \(k > r\) are equivalences (will come back to a definition of this).

We could inductively define an \((n + 1, r + 1)\)-category to be any category enriched in a category of \((n, r)\)-categories. In particular an \((\infty, 1)\)-category is any category enriched in a category of \((\infty, 0)\)-categories.

Examples 1.1.

- a \((1, 1)\)-category is an ordinary category
- a \((1, 0)\)-category is a groupoid
- an “\(\infty\)-category” generally refers to an \((\infty, 1)\)-category, that is a category with higher morphisms above degree \(n\) invertible.

1.4. Spaces are \(\infty\)-groupoids.

Definition 1.2. An \(\infty\)-groupoid is an \((\infty, 0)\)-category.

For example, any topological space canonically determines an \((\infty, 0)\)-groupoid as follows:

- 0-morphisms (objects) points
- 1-morphisms directed paths between points
- 2-morphisms homotopies of paths
- 3-morphisms homotopies of homotopies
  
  

1.5. Various models. Referring to something as an “infinity-category” is a bit loaded. In order to get a good handle on infinity categories, we should have some type of model of what a good theory of infinity categories should look like.
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\textit{K-theory of infinity categories}  

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In particular a model should be a home for infinity categories, i.e. a category whose objects are infinity categories. In our previous example, for instance, we saw that Top was a good model for \( \infty \)-groupoids.

Here are a few models of \((\infty, 1)\)-categories:

- quasi-categories
- simplicially enriched categories
- topologically enriched categories
- Segal categories
- complete Segal spaces

These are all “equivalent” in the sense that they have model structures and Quillen equivalences between them.

For the remainder of this talk, an \((\infty, 1)\)-category will mean a quasi-category.

1.6. Quasi-categories. Recall that we have a standard \( n \)-simplex \( \Delta^n \in sSet \). Its boundary is denoted by \( \partial \Delta^n \).

**Definition 1.3.** The \textit{ith horn}, denoted \( \Lambda^n_i \), is the boundary \( \partial \Delta^n \) minus the face opposite the \( i \)th vertex.

\[
\begin{align*}
\Lambda^2_0 & \quad \{1\} \quad \{0\} \quad \{2\} \\
\Lambda^2_1 & \quad \{1\} \quad \{0\} \quad \{2\} \\
\Lambda^2_2 & \quad \{1\} \quad \{0\} \quad \{2\}
\end{align*}
\]

**Definition 1.4.** We say that a simplicial set \( X \) is a \textit{quasicategory} if any inclusion of a horn \( \Lambda^n_i \), with \( 0 < i < n \), extends to an inclusion of the \( n \)-simplex:

\[
\Lambda^n_i \rightarrow X
\]

We denote by \( q\mathbf{Cat} \) the full subcategory of \( s\mathbf{Set} \) containing all quasi-categories.
1.7. **Horn filling.** Morally, what does it mean for the dashed arrow to exist:

![Diagram](image)

Consider the smallest example: $\Lambda^2_1$.

![Diagram](image)

The inclusion of this into the simplicial set $X$ corresponds to the selection of three 0-cells (which we consider to be objects) and two composable 1-cells (which we consider to be morphisms). The existence of a “filling” of this horn (an extension to the 2-simplex) means that you can compose these 1-cells in a way that *maybe doesn’t commute strictly*, but commutes up to some 2-cell.

Analogously, filling a horn $\Lambda^n_i$ for $0 < i < n$ means that for any composable collection of $(n-1)$-morphisms in a quasi-category $X$, there is a way to compose them weakly in $X$, where the composition is witnessed by some $n$-cell.

1.8. **Nerves of categories: $\Lambda^2_1$.** Suppose $X = N(C)$ is the simplicial set obtained as the nerve of some small category $C$ (remember the nerve had as $n$-cells strings of $n$-composable morphisms in $C$).

For any inclusion $\Lambda^2_1 \to NC$, this specifies two morphisms $f$ and $g$ in $C$ which are composable. We remark that this horn can be filled uniquely by the composite $g \circ f$, and the 2-cell witnessing this composition is the identity.

1.9. **Nerves of categories: $\Lambda^3_2$.** The image of $\Lambda^3_2 \to NC$ looks like:
where the back face is missing. The bottom face exists, so the back unlabelled arrow must be equal to the composite of $h$ and $gf$, giving the arrow $h \circ (gf)$. In order to fill the back face, we must see that the back arrow commutes up to some higher cell, that is, there is a 2-cell witnessing the composite $(hg) \circ f \Rightarrow h \circ (gf)$.

However because $\mathcal{C}$ was a category, we have associativity of morphisms. Thus the back face fills, and the entire 3-cell in the center fills, corresponding to the fact that the following composites are equal:

$$h \circ g \circ f = h \circ (g \circ f) = (h \circ g) \circ f.$$

1.10. **Nerves of categories: higher horns.** As you might imagine, filling other horns in $N\mathcal{C}$ has analogous interpretations, corresponding to various ways to compose $n$-composable morphisms. Moreover since the composition is strict (higher cells witnessing this composition are the identity) we have the following result.

**Proposition 1.5.** The nerve of any small 1-category is a quasi-category; moreover, horns fill uniquely: for $0 < i < n$ we have a unique dashed map

$$\Lambda^i_n \to X$$

This condition is actually sufficient to recognize when a simplicial set arises as the nerve of a category.

**Proposition 1.6.** We have that a simplicial set $X$ is the nerve of a category $\mathcal{C}$ if and only if for all $0 < i < n$, the inclusion of any horn $\Lambda^i_n \to X$ extends uniquely to the inclusion of an $n$-simplex.

1.11. **Outer horns.** For quasi-categories, we said that we wanted filling for horns $\Lambda^i_n$ where $0 < i < n$. Why shouldn’t we expect filling for $i = 0, n$? Consider the following
example:

\[
\begin{array}{ccc}
{1} & \xrightarrow{f} & {0} \\
\downarrow & & \downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
{2} & \xrightarrow{g} & {0}
\end{array}
\]

Say we were mapping \( \Lambda^n_0 \to N\mathcal{C} \) to the nerve of a category. In order for the dashed map to exist, it must be equal to \( gf^{-1} \), that is, \( f \) must be an isomorphism in the category \( \mathcal{C} \). In general there is no way to guarantee this. However if all maps in \( \mathcal{C} \) were isomorphisms, then we would have this filling.

**Proposition 1.7.** The nerve of a groupoid admits horn filling for all \( \Lambda_i^n \), where \( 0 \leq i \leq n \).

**Definition 1.8.** If a simplicial set \( X \) admits horn filling for all \( \Lambda_i^n \) for \( 0 \leq i \leq n \), we say it is a **Kan complex**.

The category \( \text{Kan} \) of Kan complexes serves as a model for \((\infty,0)\)-categories.

1.12. **Hom-sets.** The full subcategory \( \text{qCat} \subseteq \text{sSet} \) serves as a model for \((\infty,1)\)-categories. In particular for \( C,D \in \text{qCat} \), we define an \( \infty\)-functor \( F : C \to D \) to just be any morphism in the ambient category of simplicial sets.

Let \( X \in \text{qCat} \), then for two vertices \( a,b \in X_0 \) (remember these are supposed to be objects) we should describe \( \text{Hom}_X(a,b) =: X(a,b) \). By our discussion of enrichment, we should expect this object to be a Kan complex.

Consider the source and target maps

\[
(s,t) : \text{Hom}_{\text{qCat}}(\Delta^1, X) \to \text{Hom}_{\text{qCat}}(\Delta^0 \amalg \Delta^0, X) = X \times X,
\]

and let \( X(a,b) \) denote the fiber of this map over the pair \( (a,b) \). We define this to be the hom-object \( \text{Hom}_X(a,b) \).

**Proposition 1.9.** [Lur09, p. 1.2.2.3] If \( X \in \text{qCat} \) then \( X(a,b) \in \text{Kan} \) for any \( a,b \in X_0 \).

1.13. **Adjunctions.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be quasi-categories, and let \( a \in \mathcal{C} \) and \( b \in \mathcal{D} \). We say that \( \infty\)-functors \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{C} \) are **adjoint** if we have a natural weak equivalence of Kan complexes

\[
\text{Hom}_\mathcal{D}(Fa,b) \simto \text{Hom}_\mathcal{C}(a,Gb).
\]

Here a weak equivalence of Kan complexes means a weak equivalence after geometric realization.

1.14. **Terminal objects.** Let \( \mathcal{C} \in \text{qCat} \).

**Definition 1.10.** We say that \( x \in \mathcal{C}_0 \) is **terminal** if, for every \( a \in \mathcal{C}_0 \), the Kan complex \( \mathcal{C}(a,x) \) is contractible (meaning its geometric realization is contractible). Similarly, \( x \in \mathcal{C}_0 \) is **initial** if \( \mathcal{C}(x,a) \) is contractible for all \( a \in \mathcal{C}_0 \).
We say $\mathcal{C}$ is pointed if it has a zero object, which is an object that is both initial and terminal.

In general limits and colimits are hard to construct, see Higher Topos Theory Chapter 4 for more detail.

1.15. Cofibers. Suppose that $\mathcal{C}$ is a quasi-category which has a zero object, denoted $*$, and pushouts.

**Definition 1.11.** The cofiber of a morphism $f : X \to Y$ in $\mathcal{C}$ is defined to be the pushout

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow \quad \quad \downarrow \\
* \quad \quad \text{cofib}(f)
\end{array}
\]

We refer to a sequence $X \xrightarrow{f} Y \to \text{cofib}(f)$ as a cofiber sequence.

**Definition 1.12.** The suspension of $X$ is defined to be the cofiber of the unique map $X \xrightarrow{!} *$:

\[
\begin{array}{c}
X \longrightarrow * \\
\downarrow \quad \quad \downarrow \\
* \quad \quad \Sigma X.
\end{array}
\]

2. $K_0$ for infinity categories

2.1. $K_0(\mathcal{C})$. Let $\mathcal{C}$ be a pointed $\infty$-category admitting pushouts. Then define $K_0(\mathcal{C})$ to be the free abelian group $[X]$ on objects of $\mathcal{C}$ modulo that a cofiber sequence $Z \to X \to Y$ gives the relation $[Z] + [Y] = [X]$.

**Exercise 2.1.** $K_0(\mathcal{C})$ is abelian.

**Exercise 2.2.** We have that $[*] = 0$.

**Exercise 2.3.** We have that $[\Sigma X] = -[X]$.

**Warning:** If $\mathcal{C}$ admits infinite coproducts, then any object $X$ fits into a cofiber sequence

\[
\Pi_{n \geq 1} X \to \Pi_{n \geq 0} X \to X,
\]

for which we see $[X] = 0$. 
2.2. Functoriality of $K_0(\mathcal{C})$. Suppose $\mathcal{C}$ and $\mathcal{D}$ are pointed $\infty$-categories with pushouts. What conditions do we need on a functor $F : \mathcal{C} \to \mathcal{D}$ to induce a group homomorphism $K_0(\mathcal{C}) \to K_0(\mathcal{D})$?

Clearly we need $F$ to preserve the zero object. Moreover we need that $[F(X \amalg Y)] = [F(X)] + F[Y]$, that is, since $X \to X \amalg Y \to Y$ is a cofiber diagram, so must be $F(X) \to F(X \amalg Y) \to F(Y)$. Therefore we should require $F$ to preserve cofiber sequences as well.

**Example 2.4.** We remark that $\Sigma$ was a colimit itself, thus it preserves all finite colimits. The functor $\Sigma : \mathcal{C} \to \mathcal{C}$ induces the multiplication by $(-1)$ map on $K_0(\mathcal{C})$.

2.3. Stable $\infty$-categories.

**Definition 2.5.** We say an $\infty$-category $\mathcal{C}$ is stable if it is pointed, has pushouts, and so that the endofunctor $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence of categories.

**Properties of stable $\infty$-categories:**

1. $\mathcal{C}$ has all finite limits and colimits
2. A square in $\mathcal{C}$ is a pullback square if and only it is a pushout square
3. A functor between stable $\infty$-categories preserves the zero object and cofibers if and only if it preserves all finite colimits.

**Example 2.6.** The $\infty$-category of spectra is stable.

For any category, it admits a stabilization, that is a functor to a stable infinity category, initial among such functors. This is given by the Spanier-Whitehead category $\text{SW}(\mathcal{C})$, defined as the colimit:

$$\text{SW}(\mathcal{C}) := \text{colim} \left( \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \cdots \right).$$

**Remark 2.7.** We see that $\Sigma$ preserves all colimits, therefore

$$K_0(\mathcal{C}) \simeq K_0(\text{SW}(\mathcal{C})).$$

So without loss of generality, for $K_0$, we can assume we are working with stable $\infty$-categories (this will be true in general).

3. Constructing higher $K$-theory

3.1. **Reminder: $K$-theory of a Waldhausen category.** Briefly, we had a category $\mathcal{C}$ with cofibrations and weak equivalences.

- we built categories $S_n \mathcal{C}$, whose objects were these “inverted staircase” diagrams of pushouts
- this gave a bisimplicial set $S \mathcal{C}$, for which we could take any type of geometric realization, which all yielded equivalent spaces
- the fundamental group of this space was $K_0(\mathcal{C})$, which is shifted from what we want, so we loop the space to define $K(\mathcal{C})$. 


3.2. **Waldhausen K-theory of ∞-categories.** **Goal:** to replicate the construction of Waldhausen $K$-theory for ∞-categories in order to define the higher $K$-theory of ∞-categories.

This will proceed as follows:

- starting with an ∞-category $\mathcal{C}$, we get the abelian group $K_0(\mathcal{C})$
- build categories $S_n \mathcal{C}$, which under the nerve functor are considered as ∞-categories
- take the geometric realization of this bisimplicial set
- again, take the loop space to arrive at $K(\mathcal{C})$

3.3. **Objects as paths: 2-simplices.** As in the Waldhausen construction for Waldhausen categories, we want to build a based space $W$, where each $[X] \in K_0(\mathcal{C})$ corresponds to a path $p_X$ in $W$ beginning and ending at the base point $\ast$.

For a cofiber sequence $X' \to X \to X''$ we want the paths $p_{X'} \circ p_{X''}$ and $p_X$ to be homotopic, in order to encode the relations on $K_0(\mathcal{C})$ as relations in $\pi_1(W)$. That is, we need a 2-simplex:

\[ \begin{array}{ccc}
  p_{X''} & \ast & p_{X'} \\
  * & p_X & * \\
\end{array} \]

3.4. **3-simplices.** What can we say for an arbitrary pair of maps $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$, not necessarily forming a cofiber diagram?

**Proposition 3.1.** We have that

\[ [Z] = [X] + [Y/X] + [Z/Y]. \]

**Proof 1.** Use the cofiber diagrams

\[
\begin{align*}
X \to Z & \to Z/X \quad \leadsto \quad [Z] = [X] + [Z/X] \\
(Y/X) \to (Z/X) & \to (Z/Y) \quad \leadsto \quad [Z/X] = [Y/X] + [Z/Y].
\end{align*}
\]

**Proof 2.** Use the cofiber diagrams

\[
\begin{align*}
Y \to Z & \to Z/Y \quad \leadsto \quad [Z] = [Y] + [Z/Y] \\
X \to Y & \to Y/X \quad \leadsto \quad [Y] = [X] + [Y/X].
\end{align*}
\]
We can compile all of this into the following 3-simplex:

![3-simplex diagram]

* Analogous information is available for any string of composable morphisms — how do we encode this information simplicially?

### 3.5. 3-simplices, continued.

We could also encode this information via the following diagram, where we stipulate that all rectangles in sight are pushout diagrams:

![Diagram with rectangles]

* When looking for higher analogs for how to encode the cofiber relations induced by a composite $X_1 \to X_2 \to \cdots \to X_n$ of maps, we obtain the correct notion by generalizing this diagram above.
3.6. Higher simplices. Thus when building our space \( W \), we should adjoin an \( n \)-simplex for every diagram of the form

\[
\begin{array}{c}
* \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \\
\downarrow \quad \downarrow \quad \downarrow \\
* \quad \quad \quad \quad \quad \quad \quad \quad X_n/X_1 \\
\downarrow \\
\quad \quad \quad \cdots \\
\quad \quad \downarrow \\
\quad \quad \quad * 
\end{array}
\]

Let’s formalize this— let \([n]\) be the ordered set \( \{0 < 1 < \cdots < n\} \), and let

\[
[n]^{(2)} := \{(i, j) \in [n] \times [n] : i \leq j\}.
\]

Then, by associating \([n]^{(2)}\) with its nerve, which is an \( \infty \)-category, we should view our \( n \)-simplices as objects of the \( \infty \)-functor category

\[
\text{Fun}(N([n]^{(2)}), \mathcal{C}).
\]

3.7. Gapped objects. Explicitly we define an \([n]\)-gapped object of \( \mathcal{C} \) to be a functor \( X : N([n]^{(2)}) \to \mathcal{C} \) so that

1. for each \( i \in [n] \) we have that \( X(i, i) \cong * \) in \( \mathcal{C} \) is the zero object
2. for each \( i \leq j \leq k \) we have a pushout diagram

\[
\begin{array}{ccc}
X(i, j) & \longrightarrow & X(i, k) \\
\downarrow & & \downarrow \\
X(j, j) & \longrightarrow & X(j, k),
\end{array}
\]

equivalently using the previous condition, we have a cofiber sequence

\[
X(i, j) \rightarrow X(i, k) \rightarrow X(j, k).
\]

We denote by \( \text{Gap}_{[n]}(\mathcal{C}) \) the collection of all \([n]\)-gapped objects. This forms an \( \infty \)-category.

**Proposition 3.2.** The inclusion functor \( \text{Kan} \to \text{qCat} \) admits a right adjoint, which provides the largest Kan complex contained in a quasicategory.

We denote by \( S_n(\mathcal{C}) \) the largest Kan complex contained in \( \text{Gap}_{[n]}(\mathcal{C}) \).
3.8. **Face and degeneracy maps.** For \( S_2(\mathcal{C}) \subseteq \text{Gap}_{[2]}(\mathcal{C}) \), we provide the face maps:

\[
\begin{array}{c}
\text{\( \ast \)} \xrightarrow{\ast} \text{\( X \)} \xrightarrow{f} \text{\( Y \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \xrightarrow{\ast} \text{\( Y/X \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\( \ast \)} \xrightarrow{\ast} \text{\( Y \)} \xrightarrow{g} \text{\( Z \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \xrightarrow{\ast} \text{\( Z/X \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\( \ast \)} \xrightarrow{\ast} \text{\( Z \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \xrightarrow{\ast} \text{\( Z/Y \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\( \ast \)} \xrightarrow{\ast} \text{\( Y \)} \xrightarrow{g} \text{\( Z \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \xrightarrow{\ast} \text{\( Z/Y \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\( \ast \)} \xrightarrow{\ast} \text{\( X \)} \xrightarrow{g \circ f} \text{\( Z \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \xrightarrow{\ast} \text{\( Z/X \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\( \ast \)} \xrightarrow{\ast} \text{\( X \)} \xrightarrow{f} \text{\( Y \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \xrightarrow{\ast} \text{\( Y/X \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\( \ast \)} \xrightarrow{\ast} \text{\( Y \)} \xrightarrow{g} \text{\( Z \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \xrightarrow{\ast} \text{\( Z/Y \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\( \ast \)} \xrightarrow{\ast} \text{\( Y \)} \xrightarrow{g} \text{\( Z \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \xrightarrow{\ast} \text{\( Z/Y \)} \\
\downarrow \quad \downarrow \\
\text{\( \ast \)} \\
\end{array}
\]

3.9. **Degeneracy maps.** Degeneracy maps are less interesting, we simply add in an identity along the top row and add in identities horizontally going down.

3.10. **The simplicial Kan complex.** We now have a simplicial Kan complex \( S_\bullet \mathcal{C} \). Regarding this as a bisimplicial set, we can take its geometric realization. We then define the \( K \)-theory space:

\[
K(\mathcal{C}) := \Omega \left| S_\bullet \mathcal{C} \right|.
\]

**Properties:**

- if \( F : \mathcal{C} \rightarrow \mathcal{D} \) preserves finite colimits, it induces a continuous map \( K(\mathcal{C}) \rightarrow K(\mathcal{D}) \)
- the projection functors \( \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C} \) and \( \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D} \) preserve finite colimits. These maps induce a homotopy equivalence

\[
K(\mathcal{C} \times \mathcal{D}) \xrightarrow{\sim} K(\mathcal{C}) \times K(\mathcal{D})
\]
• the coproduct functor

\[ \Pi : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \]

\[(X, Y) \mapsto X \Pi Y\]

preserves colimits, and therefore ascends to a monoidal structure

\[ K(\mathcal{C}) \times K(\mathcal{C}) \to K(\mathcal{C}). \]

This turns \( K(\mathcal{C}) \) into a grouplike \( E_\infty \)-space, that is, an infinite loop space.

• The map \( \mathcal{C} \to \text{SW}(\mathcal{C}) \) induces an equivalence

\[ K(\mathcal{C}) \cong K(\text{SW}(\mathcal{C})). \]

3.11. Examples.

(1) If \( R \) is a ring, we can take \( D^b(R) \), the derived category of the ring, which can be viewed as a stable \( \infty \)-category. We have that

\[ K(D^b(R)) \cong K(R) \cong BGL(R)^+, \]

therefore we recover the algebraic \( K \)-theory of the ring.

(2) Given a scheme \( X \), we can take its category \( \text{Perf}(X) \) of perfect complexes, which has the structure of a stable \( \infty \)-category. Taking its \( K \)-theory we recover Thomason-Trobaugh \( K \)-theory.

(3) Given a topological space \( X \), its singular chains \( \text{Sing}(X) \) is a Kan complex, and therefore an \( \infty \)-category. Let \( \mathcal{C} \subseteq \text{Fun}(\text{Sing}(X), \text{Sp}) \) be the subcategory on compact objects (see nLab). Then \( K(\mathcal{C}) \cong A(X) \) is the \( A \)-theory of the space \( X \).

4. Additivity

4.1. Additivity theorem (classically). Let \( \mathcal{E}(\mathcal{C}) \) be the category whose objects are exact sequences \((A \to B \to C)\) in a Waldhausen category \( \mathcal{C} \). Then there are three functors, \( s, t, q : \mathcal{E}(\mathcal{C}) \to \mathcal{C} \) respectively picking out each of the three objects in any exact sequence.

**Theorem 4.1.** (Additivity) If \( F' \to F \to F'' \) is an exact sequence of functors between Waldhausen categories \( \mathcal{C} \to \mathcal{D} \), then \( K_n(F) = K_n(F') + K_n(F'') \).

**Proof.** We remark that giving such an exact sequence of functors is equivalent to giving a functor \( \mathcal{C} \to \mathcal{E}(\mathcal{D}) \), so we can reduce to proving the statement for the triple \((s, t, q) : \mathcal{E}(\mathcal{D}) \to \mathcal{D}\). We prove that the functor

\[ \mathcal{D} \times \mathcal{D} \to \mathcal{E}(\mathcal{D}) \]

\[(A, B) \mapsto (A \to A \Pi B \to B)\]

is a homotopy equivalence at the level of \( K \)-theory, and the result follows. \( \square \)
4.2. Additivity for $\infty$-categories. To generalize $E(\mathcal{C})$ for infinity categories, we want a category whose objects are cofiber sequences. As we can see, this is given by $\text{Gap}_2(\mathcal{C})$, whose objects we recall are diagrams

$$
\begin{array}{ccc}
* & \longrightarrow & X \\
\downarrow & & \downarrow \\
* & \longrightarrow & Z
\end{array}
$$

where $X \to Y \to Z$ is a cofiber sequence. Since a cofiber sequence is determined up to equivalence by the map $f : X \to Y$, we have an equivalence of $\infty$-categories

$$\text{Fun}(\Delta^1, \mathcal{C}) \simeq \text{Gap}_2(\mathcal{C}).$$

Let $F : \text{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ denote the map sending a functor $\Delta^1 \to \mathcal{C}$, whose image is an arrow $X \xrightarrow{\alpha} Y$, to the pair $(X, \text{cofib}(\alpha))$.

**Theorem 4.2.** (Additivity) We have that $F$ induces a homotopy equivalence

$$K(\text{Fun}(\Delta^1, \mathcal{C})) \simeq K(\mathcal{C} \times \mathcal{C}) \simeq K(\mathcal{C}) \times K(\mathcal{C}).$$

At the level of quasi-categories, we have that $F$ admits a right homotopy inverse, given by

$$\mathcal{C} \times \mathcal{C} \to \text{Fun}(\Delta^1, \mathcal{C})$$

$$(X, Y) \mapsto (X \to X \amalg Y).$$

This gives the homotopy inverse at the level of $K$-spaces.

4.3. Corollaries of additivity.

**Corollary 4.3.** Given a cofiber sequence of functors $F' \to F \to F''$ between pointed $\infty$-categories admitting finite colimits $\mathcal{C} \to \mathcal{D}$, we have that $K(F) = K(F') + K(F'')$.

**Proof.** We have three functors $s, t, q : \text{Fun}(\Delta^1, \mathcal{D}) \to \mathcal{D}$ given by taking $X \to Y$ to $X, Y$, and $Y/X$, respectively. One can easily see that $K(t) = K(s) + K(q)$.

We see that the natural transformation $F' \to F$ determines a functor $H : \mathcal{C} \to \text{Fun}(\Delta^1, \mathcal{D})$, and we can rewrite $K(F') = K(s) \circ K(H)$, $K(F) = K(t) \circ K(H)$, and $K(F'') = K(q) \circ K(H)$.

**Corollary 4.4.** We have that suspension induces a group homomorphism $K(\Sigma) : K_n(\mathcal{C}) \to K_n(\mathcal{C})$ which is multiplication by $-1$ for every $n$.

**Proof.** Apply additivity to the cofiber sequence of morphisms $\text{id} \to * \to \Sigma$. □
5. Universality (Blumberg, Gepner, Tabuada)

5.1. Overview and related results. Results in this section are from [BGT13].

We will attempt to get a handle on what type of enlightening universal property the $K$-theory of $\infty$-categories satisfies.

For example given a category $\mathcal{C}$ with some notion of short exact sequences (exact category, triangulated category, Waldhausen category), we can say that an Euler characteristic valued in an abelian group $A$ is an assignment of group elements for each isomorphism class in $\mathcal{C}$ which splits short exact sequences, that is:

$$0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0 \Rightarrow \chi(X) = \chi(X') + \chi(X'').$$

In this sense, $K_0(\mathcal{C})$ is the universal target group for Euler characteristics. We would ideally like to extend this universality results to higher $K$-theory.

5.2. Definitions and notation. Denote by $\mathbf{Cat}_{\infty}$ the category of small $\infty$-categories (e.g. quasi-categories).

Denote by $\mathbf{Cat}_{\infty}^{ex}$ the (pointed) category of small stable $\infty$-categories and exact functors (functors which preserve finite limits and colimits).

An $\infty$-category $\mathcal{C}$ is called idempotent-complete if its image under the Yoneda embedding (here the Yoneda embedding is into functors valued in spaces) is closed under retracts. We denote by $\mathbf{Cat}_{\infty}^{perf}$ the category of small idempotent-complete stable $\infty$-categories, so we have an inclusion

$$\mathbf{Cat}_{\infty}^{perf} \subseteq \mathbf{Cat}_{\infty}^{ex}.$$ 

This inclusion admits a left adjoint (Higher Topos Theory, 5.1.4.2), which we denote by $\text{Idem} : \mathbf{Cat}_{\infty}^{ex} \rightarrow \mathbf{Cat}_{\infty}^{perf}$.

5.3. Morita equivalence. Two rings $R$ and $S$ are Morita equivalent if the categories $\text{Mod}_R$ and $\text{Mod}_S$ are equivalent. This is a weaker notion than ring isomorphism, but it is enough to guarantee that the algebraic $K$-theory of $R$ and $S$ coincide:

$$K(R) \cong K(S).$$

We say two small stable $\infty$-categories $\mathcal{C}, \mathcal{D} \in \mathbf{Cat}_{\infty}^{ex}$ are Morita equivalent if $\text{Idem}(\mathcal{C})$ and $\text{Idem}(\mathcal{D})$ are equivalent, and a morphism $\mathcal{C} \rightarrow \mathcal{D}$ is a Morita equivalence if it induces an equivalence of categories $\text{Idem}(\mathcal{C}) \sim \Rightarrow \text{Idem}(\mathcal{D})$.

5.4. Exact sequences. A sequence $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ in $\mathbf{Cat}_{\infty}^{perf}$ (of small stable idempotent-complete infinity categories) is exact if:

- the composite is zero
- $\mathcal{A} \rightarrow \mathcal{B}$ is fully faithful
the induced map $B/A \to C$ is an equivalence.

A sequence is \textit{split exact} if it is exact and there exist appropriate adjoint splitting maps.

A sequence $A \to B \to C$ in $\text{Cat}^\text{ex}_\infty$ (small stable $\infty$-categories) is (split) exact if the associated sequence

$$\text{Idem}(A) \to \text{Idem}(B) \to \text{Idem}(C)$$

is (split) exact in $\text{Cat}^\text{perf}_\infty$.

5.5. \textbf{Additive and localizing invariants.} Let $E : \text{Cat}^\text{ex}_\infty \to \mathcal{D}$ be a functor to a stable presentable* $\infty$-category $\mathcal{D}$. We say $E$ is an \textit{additive invariant} if it:

\begin{itemize}
  \item inverts Morita equivalences
  \item preserves filtered colimits
  \item sends split exact sequences to cofiber sequences.
\end{itemize}

We say $E$ is a \textit{localizing invariant} if it sends all exact sequences to cofiber sequences.

Localizing invariants are additive, but the converse does not hold; a counterexample is THH, topological Hochschild homology.

5.6. \textbf{Some more notation (sorry).} Let $\text{Fun}^\text{add}(\text{Cat}^\text{ex}_\infty, \mathcal{D})$ denote the functor category of additive invariants valued in $\mathcal{D}$.

Let $\text{Fun}^L(\mathcal{C}, \mathcal{D})$ be the $\infty$-category of colimit-preserving functors.

Let $S^\infty$ denote the $\infty$-category of spectra.

5.7. \textbf{The universal additive invariant.} Let $U_{\text{add}} : \text{Cat}^\text{ex}_\infty \to \mathcal{M}_{\text{add}}$ denote the following composite, where $\mathcal{M}_{\text{add}}$ denotes the resulting category:

\begin{itemize}
  \item apply $\text{Idem} : \text{Cat}^\text{ex}_\infty \to \text{Cat}^\text{perf}_\infty$
  \item take the Yoneda embedding $y$ where presheaves are valued in the $\infty$-category of spectra $S^\infty$
  \item restrict to the subcategory of compact objects
  \item if $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is a split exact sequence, localize at maps of the form $y(\mathcal{B})/y(\mathcal{A}) \to y(\mathcal{C})$
  \item stabilize.
\end{itemize}

\textbf{Theorem 5.1.} [BGT13, pp. 6.7, 6.10] The functor $U_{\text{add}}$ is an additive invariant, and moreover is the \textit{universal additive invariant}, in the sense that, for any stable presentable $\infty$-category $\mathcal{D}$ we have an equivalence of $\infty$-categories:

$$\text{Fun}^L(\mathcal{M}_{\text{add}}, \mathcal{D}) \simeq \text{Fun}_{\text{add}}(\text{Cat}^\text{ex}_\infty, \mathcal{D}).$$

That is, every additive invariant factors through $\mathcal{M}_{\text{add}}$. 
5.8. **The application for K-theory.** We should view $\mathcal{M}_{\text{add}}$ as some category of non-commutative motives which is the receptacle for all information about additive invariants. This turns out to be enriched in spectra.

We claim that $\mathcal{S}_\infty^\omega$, the $\infty$-category of compact spectra, is a stable idempotent-complete $\infty$-category, that is, it is an element in $\text{Cat}_{\text{perf}}^\infty$. Let $\mathcal{A}$ be any other element of $\text{Cat}_{\text{perf}}^\infty$.

**Theorem 5.2.** [BGT13, p. 1.3] There is an equivalence of spectra

$$K(\mathcal{A}) \simeq \text{Map}_{\mathcal{M}_{\text{add}}} (\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega), \mathcal{U}_{\text{add}}(\mathcal{A})).$$

In particular for $n \in \mathbb{Z}$ we have an isomorphism of abelian groups

$$K_n(\mathcal{A}) \cong \text{Hom} (\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega), \Sigma^{-n}\mathcal{U}_{\text{add}}(\mathcal{A})).$$

The suspension functor $\Sigma : \mathcal{M}_{\text{add}} \to \mathcal{M}_{\text{add}}$ turns out to agree with $S_\bullet$ (BGT, 7.17).

5.9. **The universal localizing invariant.** An analogous construction may be made to obtain a universal localizing invariant

$$\mathcal{U}_{\text{loc}} : \text{Cat}_{\infty}^{\text{ex}} \to \mathcal{M}_{\text{loc}}.$$  

This category $\mathcal{M}_{\text{loc}}$ is analogously some category of non-commutative motives which receives all information about localizing invariants. Its suspension is also given by $S_\bullet$.

Analogous results to those above can be used to describe the **non-connective K-theory** of idempotent-complete stable $\infty$-categories.

5.10. **Why do we care?** Algebraic $K$-theory of $\infty$-categories was not defined in terms of universal constructions of presheaves and localizations for infinity categories, so this provides a more universal construction.

The previous result with the Yoneda lemma provides a **total classification of natural transformations from K-theory to other additive (or localizing) invariants**.

This construction provides a tractable formulation of other interesting invariants, for example **topological Hochschild homology**, which is an additive invariant. Via the previous classification we can understand and characterize the trace map $K \to \text{THH}$, an active area of research (see [BGT13, §10]).

Even though topological cyclic homology TC is not an additive or localizing invariant (it doesn’t preserve filtered colimits) one can still get a better handle on the cyclotomic trace map $K \to TC$ (see [BGT13, p. 10.3]).

6. **Universality (a la Barwick)**

Results in this section are from [Bar16].
6.1. **Slogan.**

Algebraic $K$-theory is “a universal homology theory, which takes suitable higher categories as input and produces either spaces or spectra as output.”

6.2. **Definitions and notation.** For any $\infty$-category $\mathcal{C}$, we denote by $\iota \mathcal{C}$ its maximal Kan subcomplex.

A *pair of $\infty$-categories* $(\mathcal{C}, \mathcal{C}^\dagger)$ is an $\infty$-category $\mathcal{C}$ along with an $\infty$-subcategory $\mathcal{C}^\dagger$ so that

$\iota \mathcal{C} \subseteq \mathcal{C}^\dagger \subseteq \mathcal{C}$.

A morphism of $\mathcal{C}^\dagger$ is called an *ingressive morphism*.

A *functor of pairs* $(\mathcal{C}, \mathcal{C}^\dagger) \rightarrow (\mathcal{D}, \mathcal{D}^\dagger)$ is a functor $\mathcal{C} \rightarrow \mathcal{D}$ sending ingressive morphisms to ingressive morphisms.

6.3. **Examples of pairs.** For any $\infty$-category $\mathcal{C}$, there are two trivial pairs:

1. the *minimal pair*, denoted $\mathcal{C}^\flat$, which is the pair $(\mathcal{C}, \iota \mathcal{C})$, where we recall $\iota \mathcal{C}$ is the maximal Kan subcomplex
2. the *maximal pair*, denoted $\mathcal{C}^\sharp$, which is the pair $(\mathcal{C}, \mathcal{C})$.

6.4. **Waldhausen $\infty$-categories.** We say a pair $(\mathcal{C}, \mathcal{C}^\dagger)$ is a *Waldhausen $\infty$-category* if the following axioms hold:

1. $\mathcal{C}$ is pointed, and the map $0 \rightarrow X$ is ingressive for any $X$
2. pushouts of ingressive morphisms exist and are ingressive.

We define a *morphism of Waldhausen $\infty$-categories* to be any exact functor, by which we mean it:

- preserves zero objects
- sends pushouts along an ingressive morphism to pushouts along an ingressive morphism.

We think (roughly) as the $\infty$-categorical structure encoding and generalizing weak equivalences, and ingressive morphisms as encoding cofibrations.

We denote by $\text{Wald}_{\infty}$ the $\infty$-category of Waldhausen $\infty$-categories (*Barwick, §2*).

6.5. **Examples of Waldhausen $\infty$-categories.** Equipped with the minimal pair structure $\mathcal{C}^\flat = (\mathcal{C}, \iota \mathcal{C})$, we have a Waldhausen $\infty$-category if and only if $\mathcal{C}$ is a contractible Kan complex.

With the maximal pair structure $\mathcal{C}^\sharp = (\mathcal{C}, \mathcal{C})$, we have a Waldhausen $\infty$-category if $\mathcal{C}$ has a zero object and all finite colimits.
Any stable $\infty$-category equipped with the maximal pair structure is a Waldhausen $\infty$-category.

If $(\mathcal{C}, \mathcal{C}^{\text{cof}})$ is an ordinary 1-category with cofibrations, then its nerve $(NC, N\mathcal{C}^{\text{cof}})$ is a Waldhausen $\infty$-category.

6.6. **Theories.** A functor of $\infty$-categories is *reduced* if it sends the zero object to the terminal object.

Let $\mathcal{E}$ be the category $\text{kan}$ of Kan complexes (or more generally, any $\infty$-topos). Then we define a $\mathcal{E}$-valued theory to be any reduced functor

$$\phi : \text{Wald}\_\infty \to \mathcal{E}$$

which preserves filtered colimits. Denote by

$$\text{Thy}(\mathcal{E}) \subseteq \text{Fun}(\text{Wald}\_\infty, \mathcal{E})$$

the full subcategory spanned by $\mathcal{E}$-valued theories.

6.7. **Examples of theories.** The easiest example of a theory $\iota \in \text{Thy}(\mathcal{E})$ is the *interior functor* theory:

$$\iota : \text{Wald}\_\infty \to \text{kan}$$

$$(\mathcal{C}, \mathcal{C}^\iota) \mapsto \mathcal{C}^\iota,$$

sending a Waldhausen $\infty$-category to its maximal Kan subcomplex.

Give $\Gamma^{\text{op}}$, the category of finite pointed sets, a set of cofibrations given by monomorphisms of sets with disjoint basepoints. Then $N\Gamma^{\text{op}} \in \text{Wald}\_\infty$, and moreover this object *corepresents the interior functor* in the sense that

$$\text{Fun}_{\text{Wald}\_\infty} (N\Gamma^{\text{op}}, \mathcal{C}) \xrightarrow{\sim} \mathcal{C} \iota$$

for any $\mathcal{C} \in \text{Wald}\_\infty$ [Bar16, Prop. 10.5].

6.8. **Additive theories.** A theory is *additive* if it sends direct sums to products, and a few other technical axioms that are very involved to state [Bar16, pp. 7.4, 7.5]. We think about them as the correct analog, in this setting, of functors splitting exact sequences.

In some sense we would want $K$-theory to be an additive theory.

**Example 6.1.** The interior functor $\iota : \text{Wald}\_\infty \to \text{kan}$ is *not additive*.

For theories that fail to be additive, can we provide some additive approximation to them?

**Theorem 6.2.** *(Additivization)* [Bar16, p. 7.8] Any theory $\phi : \text{Wald}\_\infty \to \mathcal{E}$ admits an additivization $D\phi$. Moreover, it is computable as

$$D\phi \simeq \text{colim}_{n \to \infty} (\Omega^n \circ \Phi \circ \Sigma^n \circ y),$$

where $y$ is the map to the derived category $D(\text{Wald}\_\infty)$, and $\Phi$ is the derived functor of $\phi$. 
In the sense of Goodwillie calculus, this is the linearization of the functor $\iota$.

6.9. **Algebraic $K$-theory of Waldhausen $\infty$-categories.** Huge definition/theorem:

The algebraic $K$-theory functor

$$K : \text{Wald}_\infty \to \text{Kan}$$

is defined to be the additivization of the interior functor $\iota : \text{Wald}_\infty \to \text{Kan}$.

This admits a canonical delooping, so we may assume that $K$-theory is valued in connective spectra (see [Bar16, §7]).

6.10. **Classifying transformations out of $K$-theory.** Recall that $\iota$ was corepresented by $N^{-}\Gamma^{\text{op}}$. Combining this fact with the universal property of additivization, we obtain a classification of natural transformations from $K$-theory to any other additive theory.

**Proposition 6.3.** ([Bar16] pp. 10.2, 10.5.1) For any additive theory $\phi : \text{Wald}_\infty \to \text{Kan}$, there is a homotopy equivalence

$$\text{Map}(K, \phi) \simeq \text{Map}(\iota, \phi) = \text{Map}((\text{Fun}_{\text{Wald}_\infty}(N^{-}\Gamma^{\text{op}}, -), \phi) \simeq \phi(N^{-}\Gamma^{\text{op}}).$$

**Corollary 6.4.** ([Bar16, p. 10.5.2], Barratt-Priddy-Quillen) Applying this to $\phi = K$, we get that the endomorphisms of algebraic $K$-theory are

$$\text{End}(K) = K(\Gamma^{\text{op}}) = QS^0 = \text{colim}_n \Omega^n S^n.$$

6.11. **Relation to $A$-theory.** For any $\infty$-topos $\mathcal{E}$, we can take its $\infty$-category of pointed compact objects $\mathcal{E}^\omega_*$. Its algebraic $K$-theory

$$K(\mathcal{E}^\omega_*)$$

is called the $A$-theory of $\mathcal{E}$.

For $X \in \text{Kan}$, we have an $\infty$-topos $\text{Fun}(X, \text{Kan})$, and we have that

$$K(\text{Fun}(X, \text{Kan})) = A(X)$$

agrees with the $A$-theory of $X$ that we have seen.

7. **Conclusion**

7.1. **Conclusion.** Infinity categories are the most general setting for a study of algebraic $K$-theory.

Universal constructions of algebraic $K$-theory provide a framework for the analysis of $K$-theory and other theories like $A$-theory.

Representability results allow a more tangible grasp of interactions between algebraic $K$-theory and other related theories like THH and TC, as well as trace maps between these.
References

