Excess Intersection in Enumerative Geometry

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What do we study?

We are interested in counting intersections of algebraic structures, when the expected answer is finite.

Examples

How many lines are there on a cubic surface?
General set up of problems

Definition
An $n$-dimensional complex projective space is $\mathbb{C}^{n+1} \setminus \{0\}/ \sim$ where $[x_0, x_1, \ldots, x_n] \sim [\lambda x_0, \lambda x_1, \ldots, \lambda x_n]$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.

Definition
A homogeneous degree-$n$ polynomial is a polynomial with only degree-$n$ terms.
Bezout’s Theorem

In $\mathbb{CP}^n$, if $X_i$ is the zero locus of a degree-$d_i$ homogeneous polynomial for $1 \leq i \leq n$. If $\cap_i X_i$ is finite, then

$$\text{number of intersections} = d_1 \ldots d_n$$
In fact, it turns out that we might have over-counted.
Complex Vector Bundle

Definition
A \( n \)-dim complex vector bundle \( E \to X \) is an assignment of \( n \)-dim complex vector space to every \( x \in X \), in a continuous way.

Definition
A section is a function from \( X \to E \), which associates a vector in the fiber to every point in \( X \), in a continuous way.

Remark
If we let the section be homogeneous polynomials that define our algebraic structures, then the intersection points exactly correspond to the zero loci of the section.
Suppose $Z(s) = \bigcup_i Z_i$. Then,

$$e(X) = \sum_i \text{index}_{Z_i}(s)$$
Poincaré Hopf Theorem

**Theorem**

Suppose $E \rightarrow X$ is a rank $n$ oriented vector bundle. Suppose $s$ is a section such that $Z(s)$ is a collection of isolated points, then

$$e(X) = \sum_{p \in Z(s)} \deg_p(s)$$
Normal bundle is defined to be tangent directions in $X$ that don’t come from tangent spaces in $Z$.

\[ 0 \to TZ \to i^* TX \to N_{Z/X} \to 0 \]

Excess bundle is defined as the quotient $E|_Z \big/ N_{Z/X}$.
**Chern Class and Chern Polynomial**

**Definition**

The Chern classes are characteristic classes that encode information about complex vector bundles. The Chern polynomial of a rank-$n$ bundle packages information of chern classes into a polynomial

\[ c_H(E) = 1 + c_1(E)H + \cdots + c_n(E)H^n \]

**Facts**

- \( c(A \oplus B) = c(A)c(B) \)
- \( c(T_{\mathbb{P}^n}) = (1 + H)^{n+1} \)
- \( c(O_{\mathbb{P}^2}(n)) = 1 + nH \)
- \( c_i(\text{rank-}n\ \text{bundle}) = 0 \) for \( i > n \)

We can define the index as the \( k\)-th chern class of the excess bundle, where \( k \) is the dimension of \( Z \).
Example: Two Plane Conics Containing the Same Component

Suppose we want to intersect two plane conics (zero loci of degree-2 polynomials in $\mathbb{CP}^2$) that contain the same component. Let $C_1 = Z(x_0 x_1)$, $C_2 = Z(x_0 x_2)$. Then, set theoretically,

$$C_1 \cap C_2 = [1 : 0 : 0] \cup \{[0 : a : b] \in \mathbb{CP}^2\}$$

Take the section from $X$ to $E$ to be $s = (x_0 x_1, x_0 x_2)$

$$Z(x_0 x_1, x_0 x_2) = [1 : 0 : 0] \cup \mathbb{P}^1$$
Charts

\[
\mathbb{CP}_2
\]

- \( \chi_0 \)
- \( \chi_1 \)
- \( \chi_2 \)

\( \mathbb{CP}_2 \)

- \( U_0 \)
- \( U_1 \)
- \( U_2 \)

- \( [1:0:0] \)

- \( \mathbb{Z}(\chi_0 \chi_1) \)
- \( \mathbb{Z}(\chi_0 \chi_2) \)
The point $[1:0:0]$ is a transverse intersection, so it contributes 1 to the answer.

$$\text{index}_{[1,0,0]}(s) = 1$$
The Chern polynomial of the normal bundle:

\[ c(N_{\mathbb{P}^1/\mathbb{P}^2}) = \frac{c(T_{\mathbb{P}^2})}{c(T_{\mathbb{P}^1})} = \frac{(1 + H)^3}{(1 + H)^2} = 1 + H \]

The Chern polynomial of \( E \):

\[ c(E) = c(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(2)) = c(\mathcal{O}_{\mathbb{P}^2}(2))^2 = (1 + 2H)^2 = 1 + 4H + 4H^2 \]

Then, the Chern polynomial of the excess bundle:

\[ c(F) = \frac{c(E)}{c(N_{Z/X})} = (1 + 4H + 4H^2)(1 - H) = 1 + 3H \]

\[ \implies c_1(F) = \text{index}_Z(s) = 3 \]
Sheldon Katz, Enumerative Geometry and String Theory