Introduction to Lie Groups, Lie Algebra, and Representation Theory

Dennica Mitev

University of Pennsylvania

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Key Definitions, Matrix Lie Group

- **Matrix Lie Group**: Any subgroup G of $GL(n;\mathbb{C})$ with the property that if A_m is any sequence of matrices in G and A_m converges to some matrix A, then either $A \in G$ or A is not invertible.
- Call a matrix Lie group **compact** if:
  - For any sequence A_m in G where A_m converges to A, A is in G
  - There exists ad constant C such that for all $A \in G, |A_{ij} \leq C|$ for all $1 \leq i, j \leq n$
- Call a matrix Lie group G **simply connected** if it is connected and every loop in G can be shrunk continuously to a point in G.

Key Definitions, Lie Algebra

- For G a matrix Lie group, its **Lie Algebra** denoted $\mathfrak{g}$ is the set of all matrix X such that e^{tX} is in G for all real numbers t
- The definition of matrix exponential we'll be using:

$$e^x := \sum_{m=0}^{\infty} \frac{X^m}{m!}$$

- For $n \times n$ matrices A, B define **commutator** of A, B as:

$$[A, B] := AB - BA$$

- The **adjoint mapping** for each $A \in G$ is the linear map $Ad_A : \mathfrak{g} \to \mathfrak{g}$ defined by the formula

$$Ad_A(X) = AXA^{-1}$$

Baker-Campbell-Hausdorff (BCH) Formula

What we want is to be able to express the group product for a matrix Lie group completely in terms of its Lie algebra.

Theorem For all $n \times n$ complex matrices X and Y with $||X||$ and $||Y||$ sufficiently small,

$$\log(e^X e^Y) = X + \int_0^1 g(e^{ad_X} e^{tad_Y})(Y)dt$$

where

$$g(A) := \sum_{m=0}^{\infty} a_m (A - I)^m$$

Now we can easily go from elements of one to the elements of the other!

Defining Representations

Call a **finite-dimensional complex representation** (f.d.c.r) of G the Lie group homomorphism $\Pi : G \to GL(n;\mathbb{C})$. Likewise, a f.d.c.r. of $\mathfrak{g}$ is the Lie algebra homomorphism $\pi : \mathfrak{g} \to gl(n;\mathbb{C})$.

Think of a representation as a linear **action** of a group or Lie algebra on some vector space V. Then, say that a subspace W of V is **invariant** if $\Pi(A)w \in W$ for all $w \in W$ and for all $A \in G$. Likewise, a representation is **irreducible** if it has no invariant subspaces other than $W = \{0\}$ and $W = V$.

If there exists an isomorphism between two representations, then they are **equivalent**.

Generating Representations

For Π a Lie group representation on G, its associated Lie algebra representation can be found by

$$\Pi(e^X) = e^{\pi(X)}$$

for all $X \in \mathfrak{g}$.
Can generate representations in three main ways:

- **Direct Sums**
- **Tensor Products**
- **Dual Representations**

Definitions for Constructions

We say that $\mathfrak{g}$ is **indecomposable** if $\mathfrak{g}$ and $\{0\}$ are its only subalgebras such that $[X, H] \in \mathfrak{h}$ for $X \in \mathfrak{g}$ and $H \in \mathfrak{h}$. Then, we call $\mathfrak{g}$ **simple** if $\mathfrak{g}$ is indecomposable and dim $\mathfrak{g} \geq 2$. Further, we say that $\mathfrak{g}$ is **semisimple** if it is isomorphic to a direct sum of simple Lie algebras.

For a complex semisimple Lie algebra $\mathfrak{g}$, we then say that $\mathfrak{h}$ is a **Cartan subalgebra** of $\mathfrak{g}$ if:

- For all $H_1, H_2 \in \mathfrak{h}, [H_1, H_2] = 0$
- For all $X \in \mathfrak{g}$, if $[H, X] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$
- ad_H is diagonalizable

Roots and Root Spaces

We say something is a **root** of $\mathfrak{g}$ relative to Cartan subalgebra $\mathfrak{h}$ if its a nonzero linear functional α on $\mathfrak{h}$ such that $\in \mathfrak{g}, X \neq 0$ with

$$[H, X] = \alpha(H)X$$

We then say that the **root space** $\mathfrak{g}_\alpha$ is the space of all X for which $[H, X] = \alpha(H)X$ for all $H \in \mathfrak{h}$. Similarly, an element of $\mathfrak{g}_\alpha$ is a **root vector**, and we can define a respective inner product.

This just describes the eigenspace for $\mathfrak{g}$!

Visualizing the Root Space

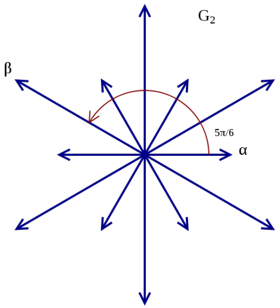

Figure 1: General Root System.

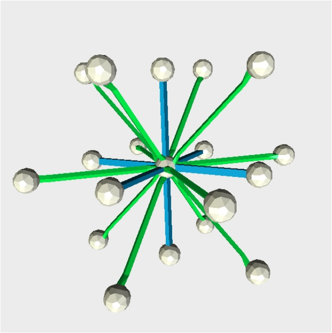

Figure 2: B3 Root System.

Weights

To generalize these roots to the inner product space containing them, we look to weights.

For π a f.d.r of $\mathfrak{g}$ on a vector space V, we say that $\mu \in \mathfrak{h}$ is a **weight** for π if $v \in V, v \neq 0$ such that

$$\pi(H)v = \langle \mu, H \rangle v$$

for all $H \in \mathfrak{h}$. Say that v is a **weight vector** for a specific weight μ, and the set of all weight vectors with weight μ is the **weight space**. The dimension of the weight space is the **multiplicity** of the weight.

Important Consequences

We can further classify a weight as a **dominant integral element** if $2\frac{\langle \mu,\alpha \rangle}{\langle \alpha,\alpha \rangle}$ is a non-negative integer for each α in the basis of our inner product space.

Theorem of the Highest Weight

- Every irreducible representation has highest weight
- Two irreducible representations with the same highest weight are equivalent
- The highest weight of every irreducible representation is a dominant integral element
- Every dominant integral element occurs as the highest weight of an irreducible representation