The Fundamental Group and Brouwer's Fixed Point Theorem

Directed Reading Project Presentation

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April 30, 2020

My Project: An Introduction to Algebraic Topology

- Book: “Algebraic Topology” by Allen Hatcher.
- Algebraic Topology: Using algebraic tools to study topological spaces.
- Goal: Assigning an algebraic structure to a topological space.

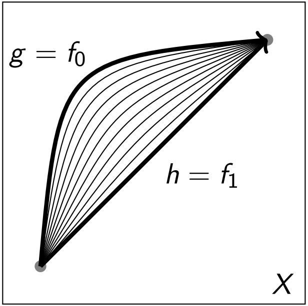
Path Homotopy

Def: A *path* in some space X is a continuous map $f : [0,1] \to X$.

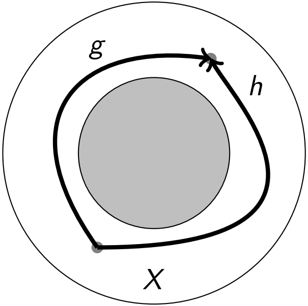
Def: A *homotopy of paths* is a family of paths $f_t : [0,1] \to X$ for $t \in [0,1]$ such that:

1. The endpoints $f_t(0)$ and $f_t(1)$ don't depend on t
2. The map defined by $F(s,t) = f_t(s)$ is continuous

Paths g and h are *homotopic* ($g \simeq h$) if there is a homotopy f_t where $f_0 = g, f_1 = h$.



homotopic: $g \simeq h$

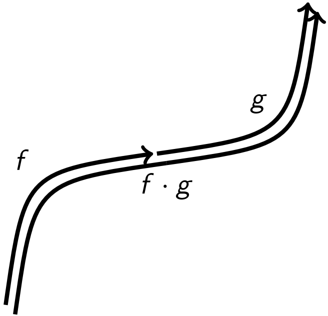


not homotopic: $g \not\simeq h$

Product Paths

Def: Given two paths $f, g : [0,1] \to X$ such that $f(1) = g(0)$, there is a *product path* $f \cdot g : [0,1] \to X$ defined by:

$$f \cdot g(s) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

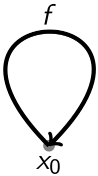


The Fundamental Group

Def: $[f]$ denotes the *homotopy class of f*, which is the set of all paths homotopic to f.

If $f \simeq g$, then $[f] = [g]$.

Def: The path f is called a *loop* when $f(0) = f(1) = x_0$.
We call x_0 the *basepoint* of f.



Def: The **fundamental group** $\pi_1(X, x_0)$ is the set of homotopy classes $[f]$ where f is a loop in X with basepoint x_0.

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Fact: $\pi_1(X, x_0)$ is a group with respect to the product $[f][g] = [f \cdot g]$:

1. Associative
2. Identity: $[c]$ where c is the constant loop i.e. $c(s) = x_0$ for any s.
3. Inverse: The inverse of $[f]$ will be $[\overline{f}]$, where $\overline{f}(s) = f(1 - s)$.

The Fundamental Group: Examples

Def: X is *path-connected* if there is a path between every pair of points.
Fact: If X is *path-connected*, then $\pi_1(X, x_0) \cong \pi_1(X, x_0')$ for any $x_0, x_0' \in X$.

Thus we can talk about $\pi_1(X)$, if X is path connected.

Ex 1 - The Plane: $\pi_1(\mathbb{R}^2) \cong 0$ (the trivial group).

For any loop f, $f \simeq c$ through a *linear homotopy:*
$f_t(s) = (1-t)f(s) + tc(s)$.
All loops are homotopic to $c \implies$ only one homotopy class

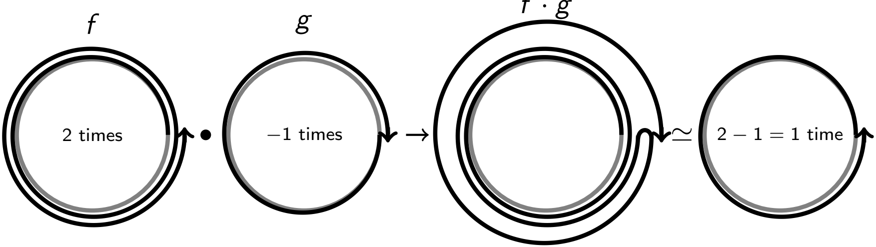
Ex 2 - The Disk: $\pi_1(D^2) \cong 0$.

Similar to Ex 1: for any loop in D^2, have linear homotopy to the constant loop.

The Fundamental Group: Examples

Ex 3 - The Circle: $\pi_1(S^1) \cong \mathbb{Z}$.

Intuition:
f loops around the circle n times
g loops around the circle m times
$f \cdot g$ loops around the circle $n + m$ times
counter-clockwise: positive, clockwise: negative



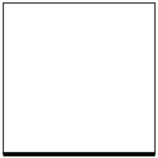
Induced Homomorphism

Def: Given a continuous map $\varphi : X \to Y$ taking basepoint $x_0 \in X$ to basepoint $y_0 \in Y$, we get an *induced homomorphism* $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $\varphi_*[f] = [\varphi \circ f]$.

Retraction

Def: For spaces $A \subset X$, a *retraction* is a continuous map $r : X \to A$ such that $r|_A = id_A$.

$X = [0,1] \times [0,1]$



$A = [0,1] \times \{0\}$

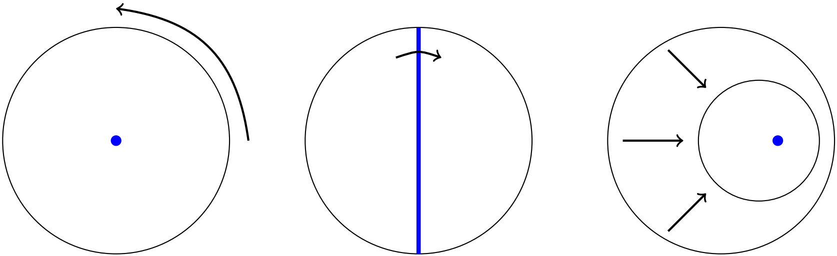
$r(x, y) = (x, 0)$ is a retraction from X to A.

Prop: Given retraction $r : X \to A$ and $x_0 \in A$, the induced homomorphism $r_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ is surjective.

Proof: For any loop f in A, f is also a loop in X and $r \circ f = f$. Thus for any $[f]_A \in \pi_1(A, x_0)$, we have that $[f]_X \in \pi_1(X, x_0)$ maps to $r_*[f]_X = [r \circ f]_A = [f]_A$.

Brouwer's Fixed Point Theorem

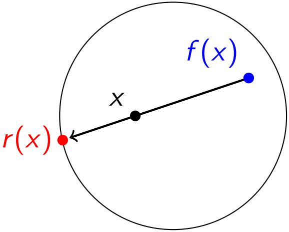
Theorem: Every continuous map $f : D^2 \to D^2$ has a fixed point, which is a point $x \in D^2$ with $f(x) = x$.



Brouwer's Fixed Point Theorem: Proof

Theorem: Every continuous map $f : D^2 \to D^2$ has a fixed point, which is a point $x \in D^2$ with $f(x) = x$.

Proof: For contradiction, suppose there was a continuous map f without any fixed points. Then, it is possible to construct map r:



$r : D^2 \to S^1$ is a retraction since it is continuous and $r|_{S^1} = id_{S^1}$. So we get a surjective group homomorphism $r_* : \pi_1(D^2) \to \pi_1(S^1)$. But it's impossible to have a surjective function $r_* : 0 \to \mathbb{Z}$. Contradiction.

Beyond

Brouwer's fixed point theorem in higher dimensions, using "higher dimensional" invariants:

- Higher homotopy groups: π_n
- Homology groups: H_n

Acknowledgements

Mona Merling
Thomas Brazelton
George Wang
Marielle Ong