Proof and Models

Mike Zhou
Logic as viewed in the past 2000 years

Classical Logic is “computationally trivial”

Logic is self-evident

Foundations to carry out math
Mathematical Logic
Logic as an Object of Mathematical Study

- We can study logic within math
- How do we characterize a proof?
- Can we exhaust all proofs?
- What does something unprovable look like?
- Is this computationally trivial?
Characterize Proofs

- Proofs can be seen as lists of sentences that refer to one another based on a set of rules.
- Starting statements will be axioms that we take for granted.
Models

• A theory is a set of sentences built on primitive formulas
• A model of that theory is a set of sentences that can be assigned a truth assignment
Examples of Models

• Natural Numbers
• Graphs
• Groups, like Integers mod 2
Semantic Truth

\[ M \models \phi \text{ if and only if } \phi \text{ holds for every instance of } M \]

• In other words, \( \phi \) is true for every truth assignment where \( M \) is satisfied
Completeness Theorem

Completeness is broken up into two parts:

• **Soundness**: \( M \vdash \phi \rightarrow M \vDash \phi \)
• **Adequacy**: \( M \vDash \phi \rightarrow M \vdash \phi \)
Change of Quantifiers

• $M \vdash \phi$ means that there exists a proof $\phi$ of from $M$
• $M \models \phi$ means that all truth assignments of $M$ satisfy $\phi$

It is generally easier to prove existence than universal properties
Consistency

• Something that is extremely hard to prove using proof theory can be intuitively expressed within a model

$$PA \not\vdash \bot \iff PA \not\models \bot$$
Consistency of Peano Arithmetic

- $\mathsf{PA} \models \bot$ if and only if every model $M$ of Peano Arithmetic is unsatisfiable
- $\mathsf{PA} \not\models \bot$ if and only if there exists some satisfiable model of Peano Arithmetic
The Natural Numbers
Unprovability $\equiv$ Countermodel

$ZF \not\vdash C \iff ZF \not\models C$

To prove unprovability of choice, we only need to find a model that satisfies $\{ZF\} \cup \neg\{C\}$

Similarly prove unprovability of negation of choice, we need to find a model that satisfies $ZFC$
Philosophically

• Philosophically, this means that everything that is unprovable is unprovable for a reason

• It’s because there exists a model that satisfies the hypotheses but not the statement
Corollary of Compactness

• Completeness helps us utilize the finite property of proofs

We state that a set of well-formed formulas $S$ tautologically implies a statement $\varphi$ (or $S \models \varphi$) if and only if every truth assignment that satisfies $S$ also satisfies $\varphi$. Prove that if $\Sigma$ is a set of well-formed formulas such that $\Sigma \models \varphi$, then there exists some finite subset $\Sigma_0$ of $\Sigma$ such that $\Sigma_0 \models \varphi$. 
Using Completeness

• If $\Sigma \models \varphi$ then $\Sigma \vdash \varphi$

• Proofs are finite, so take the finite subset of $\Sigma$ that proves $\varphi$

• Since this proves $\varphi$, it also semantically implies $\varphi$
Proofs using Finitely Many Hypotheses

• If we have finitely many hypotheses, we can deterministically determine if a set proves a statement
• Computationally trivial: Just use truth tables
Gödel’s Incompleteness Theorem

- Peano Arithmetic is Incomplete
Inspiration

“This statement cannot be proven”

• If it’s true, it can’t be proven
• If it’s false, it can be proven, meaning that if it cannot be proven, it is true
• Thus, this statement is true if and only if it cannot be proven

Issue: We can’t write this with symbols in formal logic
Gödel Coding

• An injective mapping from statements to numbers
• Uniqueness, so why not Fundamental Theorem of Arithmetic
Statement

- “The formula with Gödel Number ___ cannot be proven”
- We can show that ___ exists such that it points to itself.
Gödel’s Second Incompleteness Theorem

• No system can prove its own consistency
Classifying Simple Logic Systems
Thank you