Fun with the Fundamental Group Functor
Directed Reading Program

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Homotopy

• A **homotopy** between two continuous functions $f$ and $g$ from a topological space $X$ to a topological space $Y$ is defined to be a continuous function $H : X \times [0, 1] \rightarrow Y$ such that for all $x \in X$
  • $H(x, 0) = f(x)$ and
  • $H(x, 1) = g(x)$.

• Intuition: deforming one function into another

• For Spaces $X$ and $Y$, having a homotopy from $X$ to $Y$ is an **equivalence relation** on the set of continuous function from $X$ to $Y$.

**Figure 1: Homotopy**
Given two topological spaces $X$ and $Y$, a **homotopy equivalence** between $X$ and $Y$ is a pair of continuous maps $f : X \to Y$ and $g : Y \to X$, such that $g \circ f$ is **homotopic** to the identity map $Id_X$ and $f \circ g$ is **homotopic** to $Id_Y$.

Intuition: homotopy equivalent spaces are spaces that can be deformed continuously into one another.

**Figure 2:** Homotopy Equivalence
Path Homotopy

- Path: a **path** in a topological space $X$ is a continuous function $f : [0, 1] \rightarrow X$ with initial point $f(0)$ and terminal point $f(1)$.
- A **homotopy of paths** from $f$ to $g$ is a family $H : [0, 1] \times [0, 1] \rightarrow X$ such that
  - The endpoints $H(0) = x_0$ and $H(1) = x_1$ are independent of $t$
  - $H(s, 0) = f(s), H(s, 1) = g(s), H(0, t) = x_0,$ and $H(1, t) = x_1$
- Intuition: continuously deforming a path when keeping its endpoints fixed.

**Figure 3:** Path Homotopy
Given two paths $f, g : [0, 1] \to X$ such that $f(1) = g(0)$, there is a concatenation of path $f \cdot g$ that traverses first $f$ then $g$

$$(f \cdot g)(s) = \begin{cases} 
  f(2s) & 0 \leq s \leq \frac{1}{2} \\
  g(2s - 1) & \frac{1}{2} \leq s \leq 1. 
\end{cases}$$

Figure 4: $(f \cdot g)(s)$
Fundamental Group

- **Loops** are paths $f : [0, 1] \to X$ with the same starting and ending point $f(0) = f(1) = x_0$, and their common starting and ending point $x_0$ is called the **basepoint**.
- The **fundamental group** is the sets of path homotopy classes of the set of all loops, denoted $\pi_1(X, x_0)$.
  - Basepointed topological spaces $\to$ Groups
  - Multiplication is concatenation of paths
  - Basepoint preserving continuous functions $\to$ Group homomorphisms
  - if $f$ and $g$ are homotopic they give the same group homomorphism
Examples of the Fundamental Group

- Example 1: $\pi_1(S^1, x_0) \cong \mathbb{Z}$
  - Clockwise positive, anti-clockwise negative

- Example 2: $\pi_1(D^2, x_0) \cong \{0\}$
  - Intuition: $D^2$ is homotopy equivalent to a point so they have isomorphic fundamental groups. There's only one function from $[0, 1]$ to a point so $\pi_1(D^2, x_0) \cong \{0\}$
A Category is a collection of objects \( \{X, Y, Z\ldots\} \) and morphisms \( \{f, g, h\ldots\} \) between objects. For a pair of objects \( X \) and \( Y \) we have a collection of morphisms \( \{f, g, h,\ldots\} \) from \( X \) to \( Y \) so that

- Each object has a designated identity morphism \( \text{Id}_X : X \to X \)
- For any pair of morphisms \( f, g \) where \( f : X \to Y \) and \( g : Y \to Z \), we have \( g \circ f : X \to Z \).

A category is also subject to the following two rules:

- For any \( f : X \to Y \), we have \( \text{Id}_Y \circ f = f = f \circ \text{Id}_X \)
- Compositions are associative: \( (g \circ h) \circ f = g \circ (h \circ f) \)

Figure 5: The Category [2]
### More Examples of Categories

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Functors

• A **Functor** $F$ is a mapping $F : C \rightarrow D$ that relates two categories $C$ and $D$ such that it associates
  • Each $x \in \text{Obj}(C)$ to a $F(x) \in \text{Obj}(D)$
  • Each $f : x_1 \rightarrow x_2$ in $C$ to a $F(f) : F(x_1) \rightarrow F(x_2)$ in $D$.

• A Functor also satisfies the following two conditions
  • For each object $x \in \text{Obj}(C)$, $F(\text{Id}_x) = \text{Id}_{F(x)}$
  • For all morphisms $g, f \in C$, $F(g \circ f) = F(g) \circ F(f)$
An Example of functor

• The Fundamental group is a functor.
  • $\text{Top}_* \rightarrow \text{Grp}$
  • Basepoint preserving continuous functions $\rightarrow$ Group homomorphisms
  • two homotopic basepoint preserving continuous functions give the same group homomorphism
Example of using functor in algebraic topology

Want To Show: No retraction from a disc to a circle

• Intuition: one has a 'hole' in it and the other does not, so they must be different in some way.
• Alternative framing: Is there a continuous function $r : D^2 \to S^1$ that fixes the boundary?

Proof:
Let $i : S^1 \to D^2$ be the inclusion. Suppose for contradiction that $r$ exists, such that $r \circ i = Id_{S^1}$.

$$x \in S^1 \to i(x) = x, r(x) = x.$$
So now $i, r$ are morphisms in $\text{Top}_*$, and we can apply the fundamental group.

\[ \pi_1(S^1, x) \cong \mathbb{Z} \]
\[ \pi_1(D^2, x) \cong \{0\}. \]

These give us

\[ \pi_1(i) : \mathbb{Z} \to \{0\} \]
\[ \pi_1(r) : \{0\} \to \mathbb{Z}, \]

which is saying

\[ id_{\mathbb{Z}} = \pi_1(id_{S^1}) = \pi_1(r \circ i) = \pi_1(r) \circ \pi_1(i) = 0. \]

Contradiction $\implies$ the assumption that $r$ exists is false.
References

- Hatcher’s Algebraic Topology
- Riehl’s Category Theory in Context
groupofthecircle.gif
- https://www.math3ma.com/blog/what-is-a-functor-part-1
Thank you so much for listening!