Symplectic Geometry and Geometric Quantization

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Hamiltonian Mechanics: Phase Space and the Hamiltonian

Definition:

In classical mechanics the **phase space** is the set of all possible states of a system. For a mechanical system, the phase space consists of pairs $(\vec{q}, \vec{p})$ of generalized coordinates $(\vec{q})$ and generalized momenta $(\vec{p})$

Some Remarks:

- The set of all positions is called the **configuration space** of the system
- *Example:* The phase space of an unconstrained particle in moving in 3 dimensions can be represented by $(\vec{q}, \vec{p}) \in \mathbb{R}^3 \times \mathbb{R}^3 \sim \mathbb{R}^6$

Definition:

A **Hamiltonian** is a smooth function on the phase space: $\mathcal{H} : \mathcal{P} \to \mathbb{R}$

- The Hamiltonian encodes the total energy of a system

Hamiltonian Mechanics: Hamilton's Equations

Definition:

Given a Hamiltonian $\mathcal{H}$, the time evolution of a system in phase space is determined by solutions to **Hamilton's Equations:**

$$\dot{q}_i(t) = \frac{\partial \mathcal{H}}{\partial p_i}(\vec{q}, \vec{p}) \text{ and } \dot{p}_i(t) = -\frac{\partial \mathcal{H}}{\partial q_i}(\vec{q}, \vec{p})$$

- Can prove Newton's Laws $\iff$ Hamilton's equations are satisfied

Definition:

Given two smooth functions $F, G : M \to \mathbb{R}$ on the phase space, the **Poisson bracket** gives a third function $\{F, G\} = \sum_{i=1}^{N} \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right)$

- For any function $F(\vec{q}(t), \vec{p}(t), t)$ on the phase space, the Poisson bracket can be used to compute its time derivative: $\frac{dF}{dt} = \{F, \mathcal{H}\} + \frac{\partial F}{\partial t}$

Symplectic Manifolds and Hamiltonian Mechanics

Definition:

A **symplectic manifold** is a $2n$ dimensional smooth manifold M equipped with a two-form ω called the **symplectic form** which has the following properties:

- ω is closed: $d\omega = 0$
- ω is non-degenerate: if $\omega_p(v, w) = 0$ for all $v \in T_pM$, then $w = \vec{0}$.

The following theorem suggests that *all* symplectic manifolds of the same dimension are locally the same

Darboux's Theorem:

Let (M, ω) be a $2n$ dimensional symplectic manifold. For every $p \in \mathcal{U} \subset M$ there exists a chart $(\mathcal{U}, x_1, ...x_n, y_1, ..., y_n)$ about p such that $\omega = \sum_{j=1}^{n} dx_i \wedge dy_j$

Symplectic Manifolds are the Natural Setting for Classical Mechanics

- Configuration space can often be represented by a smooth manifold M
- Phase space represented by the cotangent bundle of M:

$$T^*M := \{(q,p) \mid q \in M, \;\; p \in T_q^*M\}$$

- Natural symplectic structure on T^*M with $\omega := dq \wedge dp$ which determines the Poisson Bracket $\{H,K\} = \omega(X_H, X_K)$

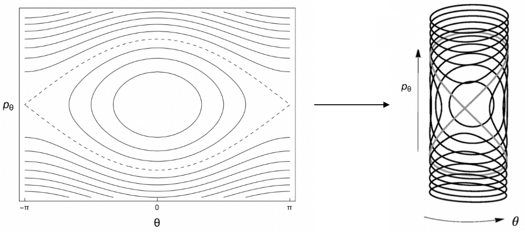

Figure: Phase space of simple pendulum is a cylinder. The configuration space is $M = \mathcal{S}^1$ which has trivial cotangent bundle, hence $T^*M = \mathcal{S}^1 \times \mathbb{R} \simeq \mathcal{C}$

Classical vs. Quantum Mechanics

- Can we use symplectic geometry to understand quantum mechanics?
  - Quantum mechanical states cannot have definite position and momenta ($\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$)

Classical Mechanics

- Set of states given by a *symplectic manifold M*
- Time evolution determined by a Hamiltonian $\mathcal{H} : M \to \mathbb{R}$ and Hamilton's equations

Quantum Mechanics

- Set of states is given by a *Hilbert Space V*
- Time evolution is determined by a linear operator $\hat{H} : V \to V$ and the Schrödinger Equation

Geometric Quantization

Kähler Manifolds and Hermitian Line Bundles

Kähler Manifolds provide the underlying structure for generalizing a classical phase space into a Hilbert space

Definition:

A **Kähler manifold** is a manifold with three mutually compatible structures:

1. a symplectic structure ω
2. a complex structure $J : T_pM \to T_pM$ (analogous to 90° rotation from multiplication by $i \in \mathbb{C}$)
3. a Riemannian metric g

such that: $\omega(v, Jw) = g(v, w)$ for all $v, w \in T_pM$.

- To geometrically quantize M need a Hermitian line bundle, L, over M with curvature $i\omega$
  - Set of all holmorphic sections of L forms a Hilbert space

Classical Spin-j Particle

- Phase space of classical spin-j particle is surface of sphere of radius j parametrized by angles (θ, ϕ) with elements representing possible angular momenta vectors:

$$\vec{j} := (j_x, j_y, j_z) = (j \sin\theta \cos\phi, j \sin\theta \sin\phi, j \cos\theta)$$

- Requiring the Poisson Bracket $\{j_k, j_l\} = \epsilon_{kl}^{m} j_m$ yields the unique symplectic form:

$$\omega = j \sin\theta d\theta \wedge d\phi = d\phi \wedge d(j \cos\theta) := dq \wedge dp$$

- How to construct Kähler manifold from (M, ω)?
    - Riemannian Structure: standard metric on a sphere of radius $\sqrt{j}$
    - Complex Structure: to get Kähler manifold the real and imaginary parts must correspond to ω and g, respectively
        - This is just the Riemann sphere $M = \mathbb{C}P^1 \sim \mathcal{S}^2$ of radius $\sqrt{j}$

Quantization of Spin

- How to equip $M = \mathbb{C}P^1$ with Hermitian L line bundle with curvature $i\omega$
  - From algebraic topology, specifically characteristic (Chern) classes, this exists iff. $\int_M \omega \in 2\pi\mathbb{Z}$

$$\int_M \omega = \int_{\mathcal{S}^2} j \sin\theta d\theta \wedge d\phi = \int_0^{2\pi}\int_0^{\pi} j \sin\theta d\theta d\phi = 4\pi j \in 2\pi\mathbb{Z} \iff j \in \frac{1}{2}\mathbb{Z}$$

  - Desired line bundle exists only for half integer values $j = 0, \frac{1}{2}, 1, \ldots$
- Holomorphic sections of L?
  - Sections about $z = 0 \in \mathbb{C}P^1$ are given by $\mathcal{F} = \{1, z, z^2, \ldots, z^N, \ldots\}$
  - $\implies$ Sections about $z = \infty \in \mathbb{C}P^1$ are given by $\mathcal{G} = \{z^{-2j}, z^{-2j+1}, \ldots, z^{-2j+N}, \ldots\}$
  - For $\mathcal{F}, \mathcal{G}$ to be regular at 0 and ∞, respectively we need $N \leq -2j$
  - Hilbert space, V, corresponds to $2j+1$ dimensional complex vector space

The Spin-$\frac{1}{2}$ Particle

- How to quantize spin-$\frac{1}{2}$ particle?
  - $j = \frac{1}{2} \implies 2j + 1 = 2$, hence we expect $\dim(V) = 2$
  - Formally, L is the dual of tautological line bundle on $M = \mathbb{C}P^1$
  - Sections are linear functionals $f : \mathbb{C}^2 \to \mathbb{C}$
  - $\implies V = \mathbb{C}^{2*} \simeq \mathbb{C}^2$
- Is $\mathbb{C}^2$ the anticipated Hilbert space?
  - From QM, arbitrary spin-$\frac{1}{2}$ particle is in state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with $\alpha, \beta \in \mathbb{C}$
  - $\implies \mathbb{C}^2$ is the Hilbert space we expect to describe spin-$\frac{1}{2}$ degree of freedom

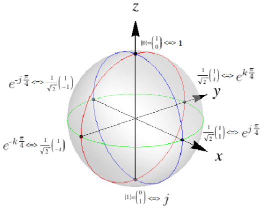

Figure: Quantum mechanical states of a spin-$\frac{1}{2}$ particle visualized using Bloch sphere

Conclusions

- Symplectic geometry is the natural setting for studying classical mechanics
  - Classical phase spaces are symplectic manifolds
- Recasting classical mechanics using symplectic geometry establishes parallels with quantum mechanics
- Geometric Quantization: Symplectic Manifold $\rightarrow$ Hilbert Space
  - Geometric quantization provides insights into quantum phenomena, such as spin

Resources/References

1. Harvard Notes on Classical Mechanics and Symplectic Geometry
2. Berkeley Notes on Quantizing Spin
3. John Baez's Notes on Geometric Quantization