Electrical Networks and Pólya’s Random Walk Theorem

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Outline of Project

Project Goals

1. Examine applications of electrical network theory to random walks
2. Classify the behavior of random walks on graphs in different dimensions ($\leq 2$ vs. $\geq 3$)

Project References

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Motivation: 1-D Random Walk

A random walker starts at node $x$ and has a $\frac{1}{2}$ probability of moving to the left/right.

From this, $p(x) = x/n$. As $n \to \infty$, $p(x) \to 0$, i.e. the random walker must return to the origin.
A random walker starts at node $x$ and has a $\frac{1}{2}$ probability of moving to the left/right.

- Probability of reaching $n$ before 0: $p(x) = \frac{1}{2}p(x-1) + \frac{1}{2}p(x+1)$, $p(0) = 0$, $p(n) = 1$
- Voltage at node $x$: $v(x) = \frac{1}{2}v(x-1) + \frac{1}{2}v(x+1)$, $v(0) = 0$, $v(n) = 1$

From this, $p(x) = x/n$. As $n \to \infty$, $p(x) \to 0$, i.e. the random walker must return to the origin.
Pólya’s Random Walk Theorem

- A walk is **recurrent** if it is certain that the random walker will return to the origin.
- A walk is **transient** if the escape probability $p_{esc} > 0$, i.e. there is a positive probability that the random walker will never return to the origin.
- (Definitions as in Doyle and Snell, modified from Pólya’s original definitions)

**Theorem**

*Simple random walks on a $d$-dimensional lattice $\mathbb{Z}^d$ are:*

- **Recurrent** for $d = 1, 2$
- **Transient** for $d \geq 3$
Random Walks on $\mathbb{Z}^2$

- Is it certain that the random walker will return to the origin? *(Recurrent)*
- Or, is there a non-zero probability that the walker will never return to the origin? *(Transient)*
Electrical network on $\mathbb{Z}^2$

- It can be shown that the escape probability $p_{esc} \propto 1/R_{eff}$, where $R_{eff}$ is the effective resistance from the origin to infinity.

- To determine $p_{esc}$ electrically, compute $R_{eff}$ between the origin and far-away grounded points.
Proof of Pólya’s Theorem for $\mathbb{Z}^2$: Shorting Nodes

- **Shorting**: Treat certain subsets of nodes as one node  
  (electrically: connect nodes with perfectly conducting wires, i.e. set the resistance of certain edges to 0)

- **Rayleigh’s Monotonicity Law**: Shorting nodes only decreases the effective resistance
Proof of Pólya’s Theorem for $\mathbb{Z}^2$: Shorting Nodes

- **Shorting**: Treat certain subsets of nodes as one node (electrically: connect nodes with perfectly conducting wires, i.e. set the resistance of certain edges to 0)

- **Rayleigh’s Monotonicity Law**: Shorting nodes only decreases the effective resistance

- **Goal**: To prove that random walks on $\mathbb{Z}^2$ are recurrent, i.e.

  $$p_{esc} \propto \frac{1}{R_{eff}} = 0 \iff R_{eff} = \infty$$

- **Technique**: Short nodes on $\mathbb{Z}^2$ such that:

  $$R_{eff} \geq R_{shorted} = \infty$$
Proof of Pólya’s Theorem for $\mathbb{Z}^2$

(Shorted nodes in red)

4 edges

$\Omega$
Proof of Pólya’s Theorem for $\mathbb{Z}^2$

(Shorted nodes in red)

4 edges

12 edges

$\frac{1}{4} \Omega$

$\frac{1}{12} \Omega$
Proof of Pólya’s Theorem for $\mathbb{Z}^2$

(Shorted nodes in red)

4 edges
12 edges
20 edges

\[ \frac{1}{4} \Omega \quad \frac{1}{12} \Omega \quad \frac{1}{20} \Omega \]
Proof of Pólya’s Theorem for $\mathbb{Z}^2$

Recalling Rayleigh’s Monotonicity Law, $R_{\text{eff}} \geq R_{\text{shorted}} = \infty \sum_{n=0}^{8n+4} = \infty$

Thus, random walks on $\mathbb{Z}^2$ are recurrent!
Proof of Pólya’s Theorem for $\mathbb{Z}^2$

Recalling Rayleigh’s Monotonicity Law,

$$R_{\text{eff}} \geq R_{\text{shorted}} = \sum_{n=0}^{\infty} \frac{1}{8n+4} = \infty$$
Proof of Pólya’s Theorem for $\mathbb{Z}^2$

Recalling Rayleigh’s Monotonicity Law,

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Thus, random walks on $\mathbb{Z}^2$ are recurrent!
Proof Idea for Higher Dimensions

- **Cutting**: Removing an edge from the network (increases resistance of edge)

- **Rayleigh’s Monotonicity Law**: Cutting edges only increases the effective resistance

- **Goal**: To prove that random walks on $\mathbb{Z}^3$ are transient, i.e.

  $$p_{esc} \propto \frac{1}{R_{eff}} > 0 \iff R_{eff} < \infty$$

- **Technique**: Cut edges outside an intricate tree such that:

  $$R_{eff} \leq R_{cut} < \infty$$
Thank you for listening!

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