Introduction to Category Theory
Directed Reading Project Presentation

Adam Zheleznyak
Mentor: Andres Mejia

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My Project: Category Theory and Algebraic Topology

- Books: “Basic Category Theory” by Tom Leinster and “Algebraic Topology” by Allen Hatcher.
- Category theory first began in the 1940s with motivations from algebraic topology.
- Today, category theory finds itself throughout many areas of mathematics, formalizing certain patterns that occur even in seemingly disparate areas.
Categories

A category $\mathcal{A}$ consists of:

- **Objects:** $\text{ob}(\mathcal{A})$
- **Morphisms:** $\mathcal{A}(A, B)$ where $A, B \in \text{ob}(\mathcal{A})$
- **Composition:** Given any $f \in \mathcal{A}(A, B)$ and $g \in \mathcal{A}(B, C)$, we can obtain a unique $g \circ f \in \mathcal{A}(A, C)$
- **Identity:** There is an identity $1_A \in \mathcal{A}(A, A)$ for all $A \in \text{ob}(\mathcal{A})$

Satisfying the following properties:

- **Associativity:** For any $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$, and $h \in \mathcal{A}(C, D)$:

$$ (h \circ g) \circ f = h \circ (g \circ f) $$

- **Identity Laws:** For any $f \in \mathcal{A}(A, B)$, $f \circ 1_A = f = 1_B \circ f$
Categories: Examples

- **Set**
  - Objects: Sets
  - Morphisms: Maps

- **Grp**
  - Objects: Groups
  - Morphisms: Group homomorphisms

- **Vect**
  - Objects: Real vector spaces
  - Morphisms: Linear maps

- **Top**
  - Objects: Topological spaces
  - Morphisms: Continuous maps

- **Top**
  - Objects: Topological spaces with a specified basepoint
  - Morphisms: Basepoint-preserving continuous maps
Functors

A map between categories is called a functor. Formally, a functor \( F : \mathcal{A} \rightarrow \mathcal{B} \) consists of:

- A function \( \text{ob}(\mathcal{A}) \rightarrow \text{ob}(\mathcal{B}) \)
- A function \( \mathcal{A}(A, A') \rightarrow \mathcal{B}(F(A), F(A')) \)

Satisfying:

- \( F(f' \circ f) = F(f') \circ F(f) \)
- \( F(1_A) = 1_{F(A)} \)

Examples:

- Forgetful functor: “forgets” the structure of something e.g. \( U : \text{Top} \rightarrow \text{Set} \) where \( U(X) \) is the underlying set of the space \( X \) and \( U(f) \) is the same map as the continuous map \( f \).
- Fundamental group: \( \pi_1 \) is a functor \( \text{Top}^* \rightarrow \text{Grp} \)
### Adjoint

Take two functors in opposite directions, \( F : \mathcal{A} \to \mathcal{B} \) and \( G : \mathcal{B} \to \mathcal{A} \). We say that \( F \) is left adjoint to \( G \) and \( G \) is right adjoint to \( F \) when there is a “natural” bijection:

\[
\mathcal{B}(F(A), B) \cong \mathcal{A}(A, G(B))
\]

for any objects \( A \in \text{ob}(\mathcal{A}) \), \( B \in \text{ob}(\mathcal{B}) \).

Essentially, this says that the maps \( F(A) \to B \) are pretty much the same as the maps \( A \to G(B) \).

**Example:** It turns out that the forgetful functor \( U : \text{Top} \to \text{Set} \) has a left adjoint \( D : \text{Set} \to \text{Top} \), where \( D(S) \) is the set \( S \) with the discrete topology, i.e. all subsets are open.

\( U \) also has a right adjoint \( I : \text{Set} \to \text{Top} \), where \( I(S) \) is the set \( S \) with a trivial topology, i.e. only \( \emptyset \) and \( S \) are open.
Example of a Limit: Product

Given category $\mathcal{A}$ and objects $X, Y$, a product of $X$ and $Y$ consists of an object $P \in \text{ob}(\mathcal{A})$ and maps

$$
\begin{array}{c}
\text{P} \\
\downarrow \quad \downarrow \\
\text{X} & \quad & \text{Y} \\
\end{array}
$$

such that for all objects $A$ with maps

$$
\begin{array}{c}
\text{A} \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{X} & \quad \text{A} & \quad \text{Y} \\
\end{array}
$$

there is a unique map $\bar{f} : A \rightarrow P$ such that this diagram commutes:

$$
\begin{array}{c}
\text{A} \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{P} & \quad \text{f_1} & \quad \text{f_2} \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{X} & \quad \text{Y} \\
\end{array}
$$
Example of a Limit: Product

Suppose \( A = \textbf{Set} \) (so \( X, Y \) are sets), then a limit is \( P = X \times Y \) with \( p_1, p_2 \) acting as projection maps.

This is because given any \( A \) and \( f_1, f_2 \), there is a unique map that satisfies the diagram above:

\[
\bar{f}(a) = (f_1(a), f_2(a))
\]

The fact that a unique map exists given any \( A \) and \( f_1, f_2 \) is an example of a *universal property*. 
Example of a Colimit: Pushout

Say we have $s : Z \rightarrow X$ and $t : Z \rightarrow Y$. A *pushout* is an object $P$ with maps $i_1, i_2$ such that

$$
\begin{align*}
Z & \xrightarrow{t} Y \\
\downarrow^s & \quad \downarrow^{i_2} \\
X & \xrightarrow{i_1} P
\end{align*}
$$

commutes, and so that given any

$$
\begin{align*}
Z & \xrightarrow{t} Y \\
\downarrow^s & \quad \downarrow^{f_2} \\
X & \xrightarrow{f_1} A
\end{align*}
$$

there is a unique $\bar{f}$ making this diagram commute:
Example of a Colimit: Pushout

Say we are working in $\mathbf{Set}$ and are given sets $X$, $Y$, and the inclusion maps $X \cap Y \hookrightarrow X$ and $X \cap Y \hookrightarrow Y$. We get the pushout to be $X \cup Y$ with the following diagram:

$$
\begin{array}{ccc}
X \cap Y & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & X \cup Y
\end{array}
$$

Theorem

*If $F$ is a left adjoint of $G$, then $G$ preserves limits and $F$ preserves colimits.*

Recall that $D$ is a left adjoint of $U$ where $D$ is the functor that gives sets the discrete topology. Because pushouts are an example of colimits, we have the same pushout when $X$, $Y$, $X \cap Y$, $X \cup Y$ are given the discrete topology.

- For those who know of the Seifert-van Kampen theorem, a similar idea can be used to prove/think about the theorem.
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