# Introduction to Category Theory

## Directed Reading Project Presentation

Adam Zheleznyak
Mentor: Andres Mejia

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# My Project: Category Theory and Algebraic Topology

- Books: “Basic Category Theory” by Tom Leinster and “Algebraic Topology” by Allen Hatcher.
- Category theory first began in the 1940s with motivations from algebraic topology.
- Today, category theory finds itself throughout many areas of mathematics, formalizing certain patterns that occur even in seemingly disparate areas.

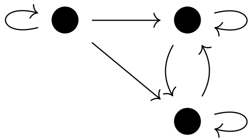
# Categories

A category $\mathscr{A}$ consists of:

- Objects: $\mathrm{ob}(\mathscr{A})$
- Morphisms: $\mathscr{A}(A, B)$ where $A, B \in \mathrm{ob}(\mathscr{A})$
- Composition: Given any $f \in \mathscr{A}(A, B)$ and $g \in \mathscr{A}(B, C)$, we can obtain a unique $g \circ f \in \mathscr{A}(A, C)$
- Identity: There is an identity $1_A \in \mathscr{A}(A, A)$ for all $A \in \mathrm{ob}(\mathscr{A})$

Satisfying the following properties:

- Associativity: For any $f \in \mathscr{A}(A, B)$, $g \in \mathscr{A}(B, C)$, and $h \in \mathscr{A}(C, D)$:
$$(h \circ g) \circ f = h \circ (g \circ f)$$
- Identity Laws: For any $f \in \mathscr{A}(A, B)$, $f \circ 1_A = f = 1_B \circ f$



# Categories: Examples

- **Set**
  - Objects: Sets
  - Morphisms: Maps
- **Grp**
  - Objects: Groups
  - Morphisms: Group homomorphisms
- $\mathbf{Vect}_{\mathbb{R}}$
  - Objects: Real vector spaces
  - Morphisms: Linear maps
- **Top**
  - Objects: Topological spaces
  - Morphisms: Continuous maps
- $\mathbf{Top}_*$
  - Objects: Topological spaces with a specified basepoint
  - Morphisms: Basepoint-preserving continuous maps

# Functors

A map between categories is called a functor.
Formally, a functor $F : \mathscr{A} \to \mathscr{B}$ consists of:

- A function $\text{ob}(\mathscr{A}) \to \text{ob}(\mathscr{B})$
- A function $\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$

Satisfying:

- $F(f' \circ f) = F(f') \circ F(f)$
- $F(1_A) = 1_{F(A)}$

Examples:

- Forgetful functor: "forgets" the structure of something e.g. $U : \mathbf{Top} \to \mathbf{Set}$ where $U(X)$ is the underlying set of the space $X$ and $U(f)$ is the same map as the continuous map $f$.
- Fundamental group: $\pi_1$ is a functor $\mathbf{Top}_* \to \mathbf{Grp}$

# Adjoints

Take two functors in opposite directions, $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$. We say that $F$ is *left adjoint* to $G$ and $G$ is *right adjoint* to $F$ when there is a "natural" bijection:

$$\mathscr{B}(F(A), B) \cong \mathscr{A}(A, G(B))$$

for any objects $A \in \text{ob}(\mathscr{A})$, $B \in \text{ob}(\mathscr{B})$.

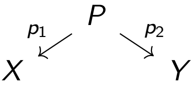
Essentially, this says that the maps $F(A) \to B$ are pretty much the same as the maps $A \to G(B)$.

Example: It turns out that the forgetful functor $U : \textbf{Top} \to \textbf{Set}$ has a left adjoint $D : \textbf{Set} \to \textbf{Top}$, where $D(S)$ is the set $S$ with the discrete topology, i.e. all subsets are open.

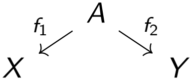
$U$ also has a right adjoint $I : \textbf{Set} \to \textbf{Top}$, where $I(S)$ is the set $S$ with a trivial topology, i.e. only $\emptyset$ and $S$ are open.

# Example of a Limit: Product

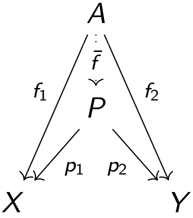
Given category $\mathscr{A}$ and objects $X, Y$, a *product* of $X$ and $Y$ consists of an object $P \in \mathrm{ob}(\mathscr{A})$ and maps



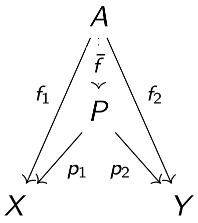

such that for all objects $A$ with maps




there is a unique map $\bar{f} : A \to P$ such that this diagram commutes:



# Example of a Limit: Product




Suppose $\mathscr{A} = \textbf{Set}$ (so $X, Y$ are sets), then a limit is $P = X \times Y$ with $p_1, p_2$ acting as projection maps.

This is because given any $A$ and $f_1, f_2$, there is a unique map that satisfies the diagram above:

$$\bar{f}(a) = (f_1(a), f_2(a))$$

The fact that a unique map exists given any $A$ and $f_1, f_2$ is an example of a *universal property*.

# Example of a Colimit: Pushout

Say we have $s : Z \to X$ and $t : Z \to Y$. A *pushout* is an object $P$ with maps $i_1, i_2$ such that

$$\begin{array}{ccc} Z & \xrightarrow{t} & Y \\ \downarrow s & & \downarrow i_2 \\ X & \xrightarrow{i_1} & P \end{array}$$

commutes, and so that given any

$$\begin{array}{ccc} Z & \xrightarrow{t} & Y \\ \downarrow s & & \downarrow f_2 \\ X & \xrightarrow{f_1} & A \end{array}$$

there is a unique $\bar{f}$ making this diagram commute:

$$\begin{array}{cccc} Z & \xrightarrow{t} & Y & \\ \downarrow s & & \downarrow i_2 & \searrow f_2 \\ X & \xrightarrow{i_1} & P & \\ & \searrow f_1 & & \overset{\bar{f}}{\searrow} \; A \end{array}$$

# Example of a Colimit: Pushout

Say we are working in **Set** and are given sets $X$, $Y$, and the inclusion maps $X \cap Y \hookrightarrow X$ and $X \cap Y \hookrightarrow Y$. We get the pushout to be $X \cup Y$ with the following diagram:

$$\begin{array}{ccc} X \cap Y & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ X & \hookrightarrow & X \cup Y \end{array}$$

## Theorem

*If $F$ is a left adjoint of $G$, then $G$ preserves limits and $F$ preserves colimits.*

Recall that $D$ is a left adjoint of $U$ where $D$ is the functor that gives sets the discrete topology. Because pushouts are an example of colimits, we have the same pushout when $X, Y, X \cap Y, X \cup Y$ are given the discrete topology.

- For those who know of the Seifert-van Kampen theorem, a similar idea can be used to prove/think about the theorem.

# Acknowledgements

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