

MATH 241 2012 SUMMER SESSION II MIDTERM

JING

JUL. 23, 1PM-3PM

The examination consists of 12 problems each worth 10 points. They are all of multiple choice, please show your work clearly and neatly, and circle your answer on the question page. No point will be given without necessary work shown. Partial or full points will be given for correct and clear work even when the final answer is wrong. No books, tables, notes, calculators, computers, phones or electronic equipment allowed.

One 8 and 1/2 inch by 11 inch paper allowed, handwritten on both sides, in your own handwriting; no substitutions of this aid allowed.

Good luck!

YOUR NAME(please print):

YOUR PENNID NUMBER:

YOUR SIGNATURE:

Don't write below this line. This table is for grading purpose only.

1		7	
2		8	
3		9	
4		10	
5		11	
6		12	
		Total	

1. How many of the following statements are true?

① If a function $f(z)$ is analytic at $z = z_0$, then $f(z)$ is differentiable at $z = z_0$.

② For a complex number $z \neq 0$, the number of different solutions to $w^n = z$ can be any number ranging from 0 to n .

③ If $f(z)$ is analytic on a domain D , then you can always find an antiderivative of $f(z)$.

④ If $f(z)$ is an entire function, then for any simple closed curve C , $\oint_C f(z)dz = 0$.

⑤ If $f(z) = \frac{P(z)}{Q(z)}$ is the fraction of two analytic functions on the domain D , then it cannot have an essential singularity on D .

⑥ A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ must either converges at all the points on the circle of convergence or diverges at all the points on the circle of convergence.

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

(F) 5

(G) 6

Proof. The answer is (D).

①: True. Analytic implies differentiable.

②: False. If $z = \rho(\cos \theta + i \sin \theta)$, $\rho \neq 0$, then there are always exactly n different n -th roots:
 $z_k = \rho^{\frac{1}{n}}(\cos(\theta + \frac{2k\pi}{n}) + i \sin(\theta + \frac{2k\pi}{n}))$, $k = 0, 1, 2, \dots, n-1$.

③: False. You need the domain to be simply connected.

④: True. Entire function is analytic everywhere, then by Cauchy-Goursat Theorem.

⑤: True. The fraction of two analytic functions can only have poles or removable singularities.

⑥: False. A power series can converge at some points on the circle of the convergence, but diverge at some other points. Example: $\sum_{k=0}^{\infty} \frac{1}{k} z^k$. □

2. For function $\frac{e^z-1}{\sin z}$, what are the singularities of $z = 0$ and $z = \pi$?
- (A) 0 is a removable singularity, π is an essential singularity
 (B) 0 is a simple pole, π is a removable singularity
 (C) 0 is a simple pole, π is a simple pole
 (D) 0 is a removable singularity, π is a removable singularity
 (E) 0 is an essential singularity, π is a simple pole
 (F) 0 is an essential singularity, π is a removable singularity
 (G) None of above

Proof. The answer is (G).

By L'hospital rule $\lim_{z \rightarrow 0} \frac{e^z-1}{\sin z} = \lim_{z \rightarrow 0} \frac{e^z}{\cos z} = 1$, so 0 is a removable singularity. At $z = \pi$, the function blows up. So let's test with $\lim_{z \rightarrow \pi} (z - \pi) \cdot \frac{e^z-1}{\sin z}$. This limit is $(e^\pi - 1) \lim_{z \rightarrow \pi} \frac{z-\pi}{\sin z}$ since $e^z - 1$ is a finite number at $z = \pi$. While $\lim_{z \rightarrow \pi} \frac{z-\pi}{\sin z} = \lim_{z \rightarrow \pi} \frac{1}{\cos z} = \frac{1}{\cos \pi} = -1$ by L'hospital rule, so this tells $z = \pi$ is a simple pole.

Therefore the answer is: 0 is a removable singularity, π is a simple pole. None of above. \square

3. Suppose $V = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ is a vector field. What are the circulation and net flux of V along the unit circle $\{z \mid |z| = 1\}$?

- (A) Circulation is 0, net flux is 0
- (B) Circulation is 2π , net flux is 0
- (C) Circulation is -2π , net flux is 0
- (D) Circulation is 2π , net flux is 2π
- (E) Circulation is 0, net flux is 2π
- (F) Circulation is 0, net flux is -2π
- (G) None of above

Proof. The answer is (E).

Let $f(z) = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$. Then $\overline{f(z)} = \frac{x-iy}{x^2+y^2} = \frac{1}{z}$. So $\oint_C f(z)dz = 2\pi i$. Therefore the circulation is the real part 0, and net flux is the imaginary part 2π . \square

4. What is the value of the integral

$$\oint_C z^{707} \cos\left(\frac{1}{z^2}\right) dz$$

where C is the circle $\{z \mid |z| = 2\}$, oriented counterclockwise?

- (A) $\frac{2\pi}{708!}i$
- (B) $-\frac{2\pi}{708!}i$
- (C) $\frac{2\pi}{354!}i$
- (D) $-\frac{2\pi}{354!}i$
- (E) $\frac{2\pi}{177!}i$
- (F) $-\frac{2\pi}{177!}i$
- (G) None of above

Proof. The answer is (D).

$\cos\left(\frac{1}{z^2}\right) = 1 - \frac{1}{2!}\left(\frac{1}{z^2}\right)^2 + \frac{1}{4!}\left(\frac{1}{z^2}\right)^4 - \dots$, so the coefficient of $\frac{1}{z^{4k}}$ is $(-1)^k \frac{1}{(2k)!}$. So the residue of $z^{707} \cos\left(\frac{1}{z^2}\right)$ should be the coefficient of $\frac{1}{z^{708}}$, which is $(-1)^{177} \frac{1}{354!} = -\frac{1}{354!}$. So the answer is $2\pi i \cdot \left(-\frac{1}{354!}\right) = -\frac{2\pi}{354!}i$. \square

5. What is the Cauchy's principal value of the improper integral?

$$P.V. \int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 4} dx$$

- (A) πi
- (B) 0
- (C) $\frac{\sqrt{3}}{6}i$
- (D) $\sqrt{3}i$
- (E) $\frac{\sqrt{3}}{3}\pi$
- (F) $\frac{\sqrt{3}}{2}\pi$
- (G) None of above

Proof. The answer is (E).

The degree of the denominator is 2, of the numerator is 0, $2 \geq 0 + 2$, so the condition on the degrees is indeed satisfied. The two poles are $-1 \pm \sqrt{3}i$ by quadratic formula, and only $-1 + \sqrt{3}i$ is on the upper half plane. So the integral $\int_{-\infty}^{\infty} \frac{1}{x^2+2x+4} dx = 2\pi i \text{Res}(f(z), -1 + \sqrt{3}i)$. This is a simple pole, so it is equal to $\frac{1}{(x^2+2x+4)'} \Big|_{x=-1+\sqrt{3}i} = \frac{1}{2(-1+\sqrt{3}i)+2} = \frac{1}{2\sqrt{3}i} = \frac{\sqrt{3}}{6i}$. Therefore the answer is $2\pi i \cdot \frac{\sqrt{3}}{6i} = \frac{\sqrt{3}}{3}\pi$. \square

6. What is the radius of convergence of the Taylor expansion for $f(z) = \frac{1}{\sinh z + i \cosh z}$ at $z = 1$?

(A) $\sqrt{\pi^2 + 1}$

(B) $\sqrt{\frac{9\pi^2}{16} + 1}$

(C) $\sqrt{\frac{\pi^2}{16} + 1}$

(D) $\sqrt{\frac{9\pi^2}{4} + 1}$

(E) $\sqrt{\frac{\pi^2}{4} + 1}$

(F) 1

(G) None of above

Proof. The answer is (C).

The singularities of $f(z)$ are just the solutions to $\sinh z + i \cosh z = 0$, i.e., $\frac{e^z - e^{-z}}{e^z + e^{-z}} = -i$. This solves out $e^{2z} = -i$, so $2z = \text{Ln}(-i) + 2n\pi i = -\frac{\pi}{2}i + 2n\pi i$, so $z = (-\frac{\pi}{4} + n\pi)i$. The closest point to 1 is $-\frac{\pi}{4}i$, and that distance is $\sqrt{\frac{\pi^2}{16} + 1}$ \square

7. What is the value of the integral

$$\int_0^{2\pi} \frac{1}{5 \sec \theta - 4} d\theta$$

where $\sec \theta := \frac{1}{\cos \theta}$?

(A) $\frac{\pi}{3}$

(B) $-\frac{\pi}{3}$

(C) $\frac{\pi}{2}$

(D) $-\frac{\pi}{2}$

(E) $\frac{4\pi}{3}$

(F) $-\frac{4\pi}{3}$

(G) None of above

Proof. The answer is (A).

Let $z = e^{i\theta}$, then $\cos \theta = \frac{z + \frac{1}{z}}{2}$, $\sec \theta = \frac{2}{z + \frac{1}{z}}$. So

$$\int_0^{2\pi} \frac{1}{5 \sec \theta - 4} d\theta = \oint_C \frac{1}{5 \cdot \frac{2}{z + \frac{1}{z}} - 4} \frac{dz}{iz} = -\frac{1}{2i} \oint_C \frac{z^2 + 1}{2z^3 - 5z^2 + 2z} dz$$

There are three singularities: $0, 2, \frac{1}{2}$, and only $0, \frac{1}{2}$ are inside the circle. So the answer is

$$-\frac{1}{2i} 2\pi i (\text{Res}(f(z), 0) + \text{Res}(f(z), \frac{1}{2})) = -\pi (\text{Res}(f(z), 0) + \text{Res}(f(z), \frac{1}{2}))$$

where $f(z) = \frac{z^2 + 1}{2z^3 - 5z^2 + 2z}$.

0 is a simple pole, so $\text{Res}(f(z), 0) = \frac{z^2 + 1}{(2z^3 - 5z^2 + 2z)'} \Big|_{z=0} = \frac{1}{2}$.

$\frac{1}{2}$ is a simple pole, so $\text{Res}(f(z), \frac{1}{2}) = \frac{z^2 + 1}{(2z^3 - 5z^2 + 2z)'} \Big|_{z=\frac{1}{2}} = \frac{\frac{1}{4} + 1}{6 \cdot \frac{1}{4} - 10 \cdot \frac{1}{2} + 2} = -\frac{5}{6}$.

Therefore the answer is $-\pi(\frac{1}{2} - \frac{5}{6}) = \frac{\pi}{3}$. □

8. If we view $w = e^{\bar{z} + \ln(2i)}$ as a mapping from z -complex plane to the w -complex plane, what is the image of upper half plane on the z -complex plane?

- (A) The region above a horizontal line
- (B) The region below a horizontal line
- (C) The region outside a circle
- (D) The region inside a circle
- (E) The region between two rays starting from the origin
- (F) A ray starting from the origin
- (G) None of above

Proof. The answer is (C).

Let $z = x + iy$, then $w = e^{i(x-iy) + \ln 2 + \frac{\pi}{2}i} = e^{(y + \ln 2) + i(x + \frac{\pi}{2})}$. So the real part $y + \ln 2$ reflects the absolute value, and the imaginary part $x + \frac{\pi}{2}$ reflects the argument. The upper half plane is $\{(x, y) | y \geq 0\}$. So this implies $|w| = e^{y + \ln 2} \geq e^{\ln 2} = 2$. So it is the outside of a circle. \square

9. Alice and Bob are playing a game. Alice choose an oriented simple closed curve C on the complex plane, but Bob does not know where the curve is or what the orientation is. Then Alice calculate the contour integral

$$\oint_C \frac{1}{z^3 + 1} dz$$

and Bob will guess what the answer is. How many times *at least* does Bob need to guess, in order to guarantee a correct answer?

- (A) 1
- (B) 3
- (C) 5
- (D) 6
- (E) 7
- (F) 8
- (G) None of above

Proof. The answer is (E).

$\frac{1}{z^3+1}$ has three poles $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, -1 , $\frac{1}{2} - \frac{\sqrt{3}}{2}i$, or in polar form $e^{\frac{\pi}{3}i}$, $e^{\pi i}$, and $e^{-\frac{\pi}{3}i}$. Since they are simple poles, so the residues are the evaluations of $\frac{1}{(z^3+1)'} = \frac{1}{3} \frac{1}{z^2}$ at those points. Use the polar form we get the residues are $\frac{1}{3}e^{-\frac{2}{3}\pi i}$, $\frac{1}{3}$, and $\frac{1}{3}e^{\frac{2}{3}\pi i}$. Explicitly, $r_1 := -\frac{1}{6} + \frac{\sqrt{3}}{6}i$, $r_2 := \frac{1}{3}$, $r_3 := -\frac{1}{6} - \frac{\sqrt{3}}{6}i$. The contour can enclose zero, one, two, or all three of them, so we can have $2\pi i r_1$, $2\pi i r_2$, $2\pi i r_3$, $2\pi i(r_1 + r_2)$, $2\pi i(r_1 + r_3)$, $2\pi i(r_2 + r_3)$, $2\pi i(r_1 + r_2 + r_3)$. Just calculate without $2\pi i$ factor is enough, so besides $r_1 = -\frac{1}{6} + \frac{\sqrt{3}}{6}i$, $r_2 = \frac{1}{3}$, $r_3 = -\frac{1}{6} - \frac{\sqrt{3}}{6}i$, we get $r_1 + r_2 = \frac{1}{6} + \frac{\sqrt{3}}{6}i$, $r_1 + r_3 = -\frac{1}{3}$, $r_2 + r_3 = \frac{1}{6} - \frac{\sqrt{3}}{6}i$, $r_1 + r_2 + r_3 = 0$. Take the orientation into consider, we still get back to these numbers. So in sum we have 7 possible answers altogether, so Bob needs to guess at least 7 times to guarantee a correct answer. □

10. Suppose $u(x, y) = x^3 + ax^2y + bxy^2 + 2y^3$ is a harmonic function, and $v(x, y)$ is its conjugate harmonic function. Moreover, we know $v(0, 0) = 1$. If we denote $c := v(1, 1)$, what are the values of a, b , and c ?

- (A) $a = -3, b = -6, c = -1$
- (B) $a = -6, b = -3, c = -1$
- (C) $a = -3, b = -6, c = 0$
- (D) $a = -6, b = -3, c = 0$
- (E) $a = -3, b = -6, c = 1$
- (F) $a = -6, b = -3, c = 1$
- (G) None of above

Proof. The answer is (B).

$\frac{\partial^2 u}{\partial x^2} = 6x + 2ay, \frac{\partial u}{\partial y^2} = 2bx + 12y$. So $0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y^2} = 6x + 2ay + 2bx + 12y$ implies $a = -6, b = -3$. So $u = x^3 - 6x^2y - 3xy^2 + 2y^3$. $\frac{\partial u}{\partial x} = 3x^2 - 12xy - 3y^2 = \frac{\partial v}{\partial y}$, so $v = 3x^2y - 6xy^2 - y^3 + f(x)$ for some function $f(x)$. Then use the other equation from the Cauchy-Riemann equations, we get $\frac{\partial u}{\partial y} = -6x^2 - 6xy + 6y^2$, therefore $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6x^2 + 6xy - 6y^2$. On the other hand, from what we know about v we get $\frac{\partial v}{\partial x} = 6xy - 6y^2 + f'(x)$. Therefore $f'(x) = 6x^2, f(x) = 2x^3 + D$, where D is a constant. So

$$v(x, y) = 3x^2y - 6xy^2 - y^3 + 2x^3 + D$$

Let $x = 0, y = 0$ we get $D = v(0, 0) = 1$. So $c = v(1, 1) = 3 - 6 - 1 + 2 + 1 = -1$. Therefore $a = -6, b = -3, c = -1$. □

11. Suppose the Laurent expansion of $\frac{1}{z(z-1)}$ over $\{z \mid 1 < |z-2| < 2\}$ is $\sum_{k=-\infty}^{\infty} a_k(z-2)^k$. What is the value $a_{-2} + a_{-1} + a_0 + a_1 + a_2$?

(A) $\frac{7}{8}$

(B) $\frac{3}{8}$

(C) $\frac{1}{8}$

(D) $-\frac{1}{8}$

(E) $-\frac{3}{8}$

(F) $-\frac{7}{8}$

(G) None of above

Proof. The answer is (D).

Partition of fractions, $\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$, get the equations $A + B = 0$, $-A = 1$. So $A = -1$, $B = 1$.

We get $\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$.

For $\frac{1}{z}$, the singularity 0 is outside the annulus, so $\frac{1}{z} = \frac{1}{(z-2)+2} = \frac{1}{2} \left(\frac{1}{1+\frac{z-2}{2}} \right) = \frac{1}{2} \left(1 - \frac{z-2}{2} + \left(\frac{z-2}{2}\right)^2 - \dots \right)$.

Take the negative sign in front into consider, so $a_0 = -\frac{1}{2}$, $a_1 = \frac{1}{4}$, $a_2 = -\frac{1}{8}$.

For $\frac{1}{z-1}$, the singularity 1 is inside the annulus, so

$$\frac{1}{z-1} = \frac{1}{z-2} \frac{z-2}{z-1} = \frac{1}{z-2} \frac{1}{\frac{(z-2)+1}{z-2}} = \frac{1}{z-2} \frac{1}{1+\frac{1}{z-2}}$$

So the Laurent expansion is going to be

$$\frac{1}{z-2} \left(1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \dots \right) = \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \dots$$

So $a_{-2} = -1$, $a_{-1} = 1$.

Therefore $a_{-2} + a_{-1} + a_0 + a_1 + a_2 = (-1) + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = -\frac{3}{8}$. □

12. What is the value of $\int_C \frac{1}{z+1} dz$, where C is the part of unit circle $\{z \mid |z| = 1\}$ from the initial point $z = 1$ to the terminal point $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, going counterclockwise?

- (A) $\frac{2\pi}{3}i$
- (B) $\frac{2\pi}{3}i - \ln 2$
- (C) $\frac{\pi}{3}i$
- (D) $\frac{\pi}{3}i - \ln 2$
- (E) 0
- (F) $\ln 2$
- (G) None of above

Proof. The answer is (D).

$\frac{1}{z+1}$ is analytic on $\mathbb{C} \setminus \mathbb{R}_{\leq -1}$, and the contour is indeed contained in this domain, so we can have an antiderivative for $\frac{1}{z+1}$ on this domain, i.e., $\text{Ln}(z+1)$. So $\int_C \frac{1}{z+1} dz = \text{Ln}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i + 1) - \text{Ln}(1+1) = \text{Ln}(\frac{1}{2} + \frac{\sqrt{3}}{2}i) - \text{Ln}(2) = \frac{\pi}{3}i - \ln 2$. \square