

ON PARTIAL REGULARITY OF STEADY-STATE SOLUTIONS TO THE 6D NAVIER-STOKES EQUATIONS

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ABSTRACT. Consider steady-state weak solutions to the incompressible Navier-Stokes equations in six spatial dimensions. We prove that the 2D Hausdorff measure of the set of singular points is equal to zero. This problem was mentioned in 1988 by Struwe [24], during his study of the five dimensional case.

1. INTRODUCTION

In this paper we consider the incompressible steady-state Navier-Stokes equations in six spatial dimensions with unit viscosity

$$(1.1) \quad u \nabla u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0,$$

in a domain $\Omega \subset \mathbb{R}^6$. We are interested in the partial regularity of weak solutions (u, p) to (1.1).

Although the problem of the global regularity of solutions to the time-dependent Navier-Stokes equations in three and higher space dimensions is still widely open, many authors have studied the partial regularity of weak solutions. In his pioneering work [19, 20, 22], Scheffer established various partial regularity results for weak solutions to the 3D Navier-Stokes equations satisfying the so-called local energy inequality. In 3D, the notion of *suitable weak solutions* was first introduced in a celebrated paper [1] by Caffarelli, Kohn and Nirenberg. They called a pair consisting of velocity u and pressure p a suitable weak solution if u has finite energy norm, p belongs to the Lebesgue space $L_{5/4}$, u and p are weak solutions to the Navier-Stokes equations, and they satisfy a local energy inequality. After proving an ε -regularity

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criteria for local boundedness of solutions, they established partial regularity of solutions and estimated the Hausdorff dimension of the singular set. They proved that, for any suitable weak solution (u, p) , there is an open subset where the velocity field u is Hölder continuous and they showed that the 1D Hausdorff measure of the complement of this subset is equal to zero. In [16], F. Lin gave a more direct and simplified proof of Caffarelli, Kohn and Nirenberg's result. A detailed treatment was later given by Ladyzhenskaya and Seregin in [15]. Recently, some extended results have been obtained in a number of papers; see Seregin [18], Gustafson, Kang and Tsai [9], and Vasseur [26], Kukavica [14], and the references therein.

Much fewer results are available in the literature for the 4D and higher dimensional time-dependent Navier-Stokes equations, in which case the problem is more super-critical. In [21], Scheffer showed that there exists a weak solution u in $\mathbb{R}^4 \times \mathbb{R}^+$, which may not necessarily verify the local energy estimate, such that u is continuous outside a locally closed set of $\mathbb{R}^4 \times \mathbb{R}^+$ whose 3D Hausdorff measure is finite. In a recent paper [2], the first author and D. Du proved that, for any local-in-time smooth solution to the 4D Navier-Stokes equations, the 2D Hausdorff measure of the set of singular points at the first potential blow-up time is equal to zero. We remark that, in terms of the method, the dimension four in [2] is critical due to the following reason. To the best of our knowledge all the existing methods on partial regularity for the Navier-Stokes equations share the following prerequisite condition: in the energy inequality the nonlinear term should be controlled by the energy norm under the Sobolev imbedding theorem. Actually, four is the highest dimension in which we have such condition: $L_3^t L_3^x \hookrightarrow L_\infty^t L_2^x \cap L_2^t H^1$. In five or higher dimensions this condition fails.

This paper concerns the partial regularity of weak solutions u to the steady-state Navier-Stokes equations (1.1). In the literature, the most relevant paper on the subject is [24] by Struwe, in which he proved the following ε -regularity result, which implies that weak solutions are regular outside a singular set of zero 1D Hausdorff measure.

Theorem 1.1 (Struwe [24]). *Let Ω be an open domain in \mathbb{R}^5 and $f \in L_q(\Omega)$ for some $q > 5/2$. There exists an absolute constant $\varepsilon_0 > 0$ such that the following holds true. If $u \in H^1(\Omega; \mathbb{R}^5)$ is a weak solution to (1.1) which satisfies a generalized energy inequality, and if for some $x_0 \in \Omega$ there is $R_0 > 0$ such that*

$$r^{-1} \int_{|x-x_0|<r} |\nabla u|^2 dx \leq \varepsilon_0, \quad \forall r \in (0, R_0),$$

then u is Hölder continuous in a neighborhood of x_0 .

The proof of Theorem 1.1 relies on some techniques for proving regularity for elliptic systems (cf. Morrey [17] and Giaquinta [12]) and estimates for the linear Stokes' system due to Solonnikov [23]. A similar approach was used before by Giaquinta and Modica in [11] to study the steady-state Navier-Stokes equations in dimensions $d \leq 4$. Because time corresponds to two space dimensions, in some sense the 5D stationary problem is similar to the 3D non-stationary problem. Therefore, dimension five is the smallest dimension for which the steady-state Navier-Stokes equations are super critical. We also note that Theorem 1.1 was improved by K. Kang [13], in which partial regularity up to the boundary was established for a smooth domain $\Omega \subset \mathbb{R}^5$. The existence of regular solutions to the steady-state Navier-Stokes in high dimensions have also attracted substantial attention. We refer the reader to [8, 25, 4, 6, 5, 7, 3] and the references therein.

In [24] Struwe raised the following interesting question: does the analogous partial regularity result hold in dimension six or higher. It seems to us that some arguments in [24] do not work in six or higher dimensions. In this paper, we give a positive answer to Struwe's question in dimension six. To be more precise, we shall prove the following regularity result with a sufficiently small constant ε_0 : Let Ω be an open set in \mathbb{R}^6 , $f \in L_{6,\text{loc}}(\Omega)$, and let u be a weak solution to (1.1) satisfying a local energy inequality (2.2). Then if for some $x_0 \in \Omega$ there exists an $R_0 > 0$ such that

$$r^{-2} \int_{|x-x_0|<r} |\nabla u|^2 dx \leq \varepsilon_0, \quad \forall r \in (0, R_0),$$

then u is Hölder continuous in a neighborhood of x_0 . In particular, it follows that the 2D Hausdorff measure of the set of singular points of the weak solution u is equal to zero.

Related to Struwe's work, our proof also uses some techniques which appeared in the study of the regularity theory for elliptic systems; but our approach is quite different from that in [24]. In particular, we do not use any estimate for the linear Stokes' systems. Roughly speaking, there are three steps in our proof. In the first step, we essentially follow the argument in [2], which in turn used some ideas in [15] and [16]. The novelties are in the second and third steps. In the second step, we choose suitable test functions in the energy inequality and then use an iteration method to establish a weak decay estimate of certain scaling invariant quantities. In the last step, we successively improve this decay estimate by a bootstrap argument, and finally we use the elliptic theory to get a good estimate of the $L_{3/2}$ norm of ∇u , which yields the Hölder regularity thanks to the Morrey lemma.

It is, however, unclear to us whether similar results hold in seven or higher dimensions. In fact, we believe that six is the highest dimension to which our approach can be applied; see Remark 3.8. Therefore, in some sense, our results are critical in terms of the dimension.

To conclude this introduction, we explain some notation used in the sequel: \mathbb{R}^d is the d -dimensional Euclidean space with a fixed orthonormal basis. A typical point in \mathbb{R}^d is denoted by $x = (x_1, x_2, \dots, x_d)$. As usual the summation convention over repeated indices is enforced. And $x \cdot y = x_j y_j = \sum_{j=1}^d x_j y_j$ is the inner product for $x, y \in \mathbb{R}^d$. The standard Lebesgue spaces are denoted by L_p ($p \geq 1$). Various constants are denoted by N in general and the expression $N = N(\dots)$ means that the given constant N depends only on the contents of the parentheses.

2. SETTING AND MAIN RESULTS

For summable functions $p, u = (u_i)$ and $\tau = (\tau_{ij})$, we use the following standard differential operators

$$\begin{aligned} u_{,i} &= D_i u, & \nabla p &= (p_{,i}), & \nabla u &= (u_{,i,j}), \\ \operatorname{div} u &= u_{i,i}, & \operatorname{div} \tau &= (\tau_{ij,j}), & \Delta u &= \operatorname{div} \nabla u. \end{aligned}$$

These are all understood in the sense of distributions.

We use the following notation for spheres and balls

$$\begin{aligned} S(x_0, r) &= \{x \in \mathbb{R}^6 \mid |x - x_0| = r\}, & S(r) &= S(0, r), & S &= S(1), \\ B(x_0, r) &= \{x \in \mathbb{R}^6 \mid |x - x_0| < r\}, & B(r) &= B(0, r), & B &= B(1). \end{aligned}$$

We also denote the mean value of a summable function as follows

$$[u]_{x_0, r} = \frac{1}{|B(r)|} \int_{B(x_0, r)} u(x) dx.$$

Here $|A|$ as usual denotes the Lebesgue measure of the set A .

Let x_0 be a given point in Ω and $r > 0$ a real number such that $B(x_0, r) \subset \Omega$. It is known that in the sense of distributions one has

$$\begin{aligned} \Delta p &= D_{ij}(u_i u_j) + \operatorname{div} f \\ &= D_{ij}((u_i - [u_i]_{x_0, r})(u_j - [u_j]_{x_0, r})) + \operatorname{div} f, & \text{in } B(x_0, r). \end{aligned}$$

This will hold for a weak solution to (1.1). Now let $\eta(x)$ be a smooth function on \mathbb{R}^6 supported in the unit ball $B(1)$, $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $\bar{B}(2/3)$. We consider the decomposition

$$(2.1) \quad p = \tilde{p}_{x_0, r} + h_{x_0, r}, \quad \text{in } B(x_0, r).$$

Above $\tilde{p}_{x_0, r}$ solves the following Poisson equation

$$\Delta \tilde{p}_{x_0, r} = D_{ij}((u_i - [u_i]_{x_0, r})(u_j - [u_j]_{x_0, r})\eta_{x_0, r}) + \operatorname{div}(f\eta_{x_0, r}),$$

where $\eta_{x_0,r}(x) \stackrel{\text{def}}{=} \eta((x - x_0)/r)$. Then $h_{x_0,r}$ is harmonic in $B(x_0, r/2)$.

We will omit the indices of \tilde{p} and h whenever there is no possibility of confusion. The following notation will be used throughout the article:

$$\begin{aligned} A(r) &= A(r, x_0) = \frac{1}{r^4} \int_{B(x_0,r)} |u|^2 dx, \\ E(r) &= E(r, x_0) = \frac{1}{r^2} \int_{B(x_0,r)} |\nabla u|^2 dx, \\ C(r) &= C(r, x_0) = \frac{1}{r^3} \int_{B(x_0,r)} |u|^3 dx, \\ D(r) &= D(r, x_0) = \frac{1}{r^3} \int_{B(x_0,r)} |p - [h]_{x_0,r}|^{3/2} dx, \\ F(r) &= F(r, x_0) = \int_{B(x_0,r)} |f|^2 dx. \end{aligned}$$

Notice these objects are invariant under the natural scaling for (1.1):

$$u(x) \rightarrow \lambda u(\lambda x), \quad p(x) \rightarrow \lambda^2 p(\lambda x), \quad f(x) \rightarrow \lambda^3 f(\lambda x).$$

We will use these quantities to study the regularity of 6D steady-state suitable weak solutions to (1.1).

We say that a pair of functions (u, p) is a suitable weak solution to (1.1) in Ω if $u \in H_{\text{loc}}^1(\Omega)$ and $p \in L_{3/2, \text{loc}}(\Omega)$ satisfy (1.1) in the weak sense and additionally the generalized local energy inequality holds for any non-negative test function $\psi \in C_0^\infty(\Omega)$:

$$(2.2) \quad 2 \int_{\Omega} |\nabla u|^2 \psi dx \leq \int_{\Omega} |u|^2 \Delta \psi + (|u|^2 + 2p)u \cdot \nabla \psi + f \cdot u \psi dx.$$

The existence of regular solutions to the Dirichlet problem of (1.1) in four dimensions have been obtained Gerhardt [8], in five dimensions by Struwe [25], and in five and six dimensions by Frehse and Ruzicka [4, 6]; see also [10, 5, 7, 3] for other related results. We observe that the regular solution constructed in [6] satisfies (2.2). On the other hand, at the time of this writing, the uniqueness problem in five or higher dimensions is still wide open unless some smallness assumption is imposed on the initial conditions.

By the Sobolev imbedding theorem, for any functions $u \in H_{\text{loc}}^1$ and $r > 0$, we have the following inequality

$$(2.3) \quad \int_{B(r)} |u|^3 dx \leq N \left(\int_{B(r)} |\nabla u|^2 dx \right)^{3/2} + Nr^{-3} \left(\int_{B(r)} |u|^2 dx \right)^{3/2}.$$

This implies that $C(r)$ is well defined for a suitable weak solution.

Next we state the main results of the article.

Theorem 2.1 (ε -regularity criterion in terms of E). *Let Ω be an open set in \mathbb{R}^6 , $f \in L_{6,loc}(\Omega)$, and suppose that the pair (u, p) is a suitable weak solution to (1.1) in Ω . Then there is a positive number ε_0 satisfying the following property. Assume that for a point $x_0 \in \Omega$ the following inequality holds:*

$$(2.4) \quad \limsup_{r \downarrow 0} E(r) \leq \varepsilon_0.$$

Then u is Hölder continuous in a neighborhood of x_0 .

Theorem 2.2 (ε -regularity criterion in terms of C , D and F). *Let Ω be an open set in \mathbb{R}^6 , $f \in L_{6,loc}(\Omega)$, and suppose that the pair (u, p) is a suitable weak solution to (1.1) in Ω . There is a positive number ε_0 satisfying the following property. Assume that for a point $x_0 \in \Omega$ and for some ρ_0 such that $B(x_0, \rho_0) \subset \Omega$ the inequality*

$$(2.5) \quad C(\rho_0) + D(\rho_0) + F(\rho_0) \leq \varepsilon_0$$

holds. Then u is Hölder continuous in a neighborhood of x_0 .

Theorem 2.3 (Partial regularity). *Let Ω be an open set in \mathbb{R}^6 , $f \in L_{6,loc}(\Omega)$, and suppose that the pair (u, p) is a suitable weak solution to (1.1) in Ω . Then the 2D Hausdorff measure of the set of singular points in Ω is equal to zero.*

These results are in the spirit of the work of Caffarelli, Kohn and Nirenberg [1], where it was proved that for any suitable weak solution u to the 3D time-dependent Navier-Stokes equations the 1D Hausdorff measure of the set of singular points is equal to zero. For 5D steady-state Navier-Stokes equations, results of this type were obtained by Struwe [24].

Remark 2.4. The assumption of the external force f in Theorems 2.1-2.3 may be relaxed. It should be possible to only assume that f in $L_{p,loc}$ for some $p \in (3, 6)$ or alternatively to assume that f is in certain Morrey spaces. However, we do not intend to find such a minimal assumption of f in this paper.

3. THE PROOF

We shall prove the main theorems in three steps.

3.1. Step 1. In the first step, we want to control A , C and D in a smaller ball by their values in a larger ball under the assumption that E is sufficiently small. Here we follow the argument in [2], which in turn used some ideas in [15] and [16]. These first few estimates do not use the equation (1.1). Furthermore, in this section we often write $C(r) = C(r, x_0)$ and similarly for A , D , E and F .

Lemma 3.1. *Suppose $\gamma \in (0, 1)$, $\rho > 0$ are constants and $B(x_0, \rho) \subset \Omega$. Then we have*

$$(3.1) \quad C(\gamma\rho) \leq N[\gamma^{-3}E^{3/2}(\rho) + \gamma^{-6}A^{3/4}(\rho)E^{3/4}(\rho) + \gamma^3C(\rho)],$$

where N is a constant independent of γ , ρ and x_0 .

Proof. Denote $r = \gamma\rho$. By using the Poincaré inequality and Cauchy's inequality, we have

$$\begin{aligned} \int_{B(x_0, r)} |u|^2 dx &= \int_{B(x_0, r)} (|u|^2 - [u^2]_{x_0, \rho}) dx + \int_{B(x_0, r)} [u^2]_{x_0, \rho} dx \\ &\leq N\rho \int_{B(x_0, \rho)} |\nabla u| |u| dx + \left(\frac{r}{\rho}\right)^6 \int_{B(x_0, \rho)} |u|^2 dx. \end{aligned}$$

This is furthermore bounded by

$$\begin{aligned} &\leq N\rho \left(\int_{B(x_0, \rho)} |\nabla u|^2 dx \right)^{1/2} \left(\int_{B(x_0, \rho)} |u|^2 dx \right)^{1/2} \\ &\quad + \left(\frac{r}{\rho}\right)^6 \int_{B(x_0, \rho)} |u|^2 dx \\ &\leq N\rho^3 A^{1/2}(\rho) \left(\int_{B(x_0, \rho)} |\nabla u|^2 dx \right)^{1/2} + \left(\frac{r}{\rho}\right)^6 \left(\int_{B(x_0, \rho)} |u|^3 dx \right)^{2/3} \rho^2. \end{aligned}$$

Due to the Sobolev inequality (2.3), we obtain

$$\begin{aligned} \int_{B(x_0, r)} |u|^3 dx &\leq N \left[\left(\int_{B(x_0, r)} |\nabla u|^2 dx \right)^{3/2} \right. \\ &\quad \left. + r^{-3} \rho^{9/2} A^{3/4}(\rho) \left(\int_{B(x_0, \rho)} |\nabla u|^2 dx \right)^{3/4} + \left(\frac{r}{\rho}\right)^6 \int_{B(x_0, \rho)} |u|^3 dx \right]. \end{aligned}$$

The conclusion of Lemma 3.1 follows immediately. \square

Lemma 3.2. *Suppose $\gamma \in (0, 1/3]$ and $\rho > 0$ are constants, and $B(x_0, \rho) \subset \Omega$. Then we have*

$$(3.2) \quad D(\gamma\rho) \leq N[\gamma^{9/2}D(\rho) + \gamma^{-3}E^{3/2}(\rho) + \gamma^{-3}F^{3/4}(\rho)],$$

where N is a constant independent of γ , ρ and x_0 .

Proof. Denote $r = \gamma\rho$. Recall the decomposition of p introduced in (2.1). By using the Calderón-Zygmund estimate and the Sobolev-Poincaré inequality, one has

$$\begin{aligned}
(3.3) \quad & \int_{B(x_0, r)} |\tilde{p}_{x_0, r}(x)|^{3/2} dx \\
& \leq N \int_{B(x_0, r)} |u - [u]_{x_0, r}|^3 dx + N \int_{\mathbb{R}^6} |\Delta^{-1} \operatorname{div}(f\eta_{x_0, r})|^{3/2} dx \\
& \leq N \left(\int_{B(x_0, r)} |\nabla u|^2 dx \right)^{3/2} + N \left(\int_{B(x_0, r)} |f|^{6/5} dx \right)^{5/4}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(3.4) \quad & \int_{B(x_0, \rho)} |\tilde{p}_{x_0, \rho}|^{3/2} dx \leq N \left(\int_{B(x_0, \rho)} |\nabla u|^2 dx \right)^{3/2} \\
& \quad + N \left(\int_{B(x_0, \rho)} |f|^{6/5} dx \right)^{5/4}.
\end{aligned}$$

Since $h_{x_0, \rho}$ is harmonic in $B(x_0, \rho/2)$, any Sobolev norm of $h_{x_0, \rho}$ in a smaller ball can be estimated by any L_p norm of $h_{x_0, \rho}$ in $B(x_0, \rho/2)$. Thus, using the Poincaré inequality, one can obtain

$$\begin{aligned}
& \int_{B(x_0, r)} |h_{x_0, \rho} - [h_{x_0, \rho}]_{x_0, r}|^{3/2} dx \\
& \leq Nr^{3/2} \int_{B(x_0, r)} |\nabla h_{x_0, \rho}|^{3/2} dx \leq Nr^{15/2} \sup_{B(x_0, r)} |\nabla h_{x_0, \rho}|^{3/2}.
\end{aligned}$$

Further using the estimates for harmonic functions, the above is

$$\begin{aligned}
(3.5) \quad & \leq N \left(\frac{r}{\rho} \right)^{15/2} \int_{B(x_0, \rho/2)} |h_{x_0, \rho} - [h_{x_0, \rho}]_{x_0, \rho}|^{3/2} dx \\
& \leq N \left(\frac{r}{\rho} \right)^{15/2} \int_{B(x_0, \rho)} |p - [h_{x_0, \rho}]_{x_0, \rho}|^{3/2} + |\tilde{p}_{x_0, \rho}|^{3/2} dx.
\end{aligned}$$

Combining (3.3), (3.4) and (3.5) together yields

$$\begin{aligned}
(3.6) \quad & \int_{B(x_0, r)} |p - [h_{x_0, \rho}]_{x_0, r}|^{3/2} dx \leq N \left(\int_{B(x_0, \rho)} |\nabla u|^2 dx \right)^{3/2} \\
& + N \left(\int_{B(x_0, \rho)} |f|^{6/5} dx \right)^{5/4} + N \left(\frac{r}{\rho} \right)^{15/2} \int_{B(x_0, \rho)} |p - [h_{x_0, \rho}]_{x_0, \rho}|^{3/2} dx.
\end{aligned}$$

Since $\tilde{p}_{x_0,r} + h_{x_0,r} = p = \tilde{p}_{x_0,\rho} + h_{x_0,\rho}$ in $B(x_0, r)$, by Hölder's inequality

$$\begin{aligned}
 (3.7) \quad & \int_{B(x_0,r)} |[h_{x_0,\rho}]_{x_0,r} - [h_{x_0,r}]_{x_0,r}|^{3/2} dx \\
 &= Nr^6 |[h_{x_0,\rho}]_{x_0,r} - [h_{x_0,r}]_{x_0,r}|^{3/2} = Nr^6 |[\tilde{p}_{x_0,\rho}]_{x_0,r} - [\tilde{p}_{x_0,r}]_{x_0,r}|^{3/2} \\
 &\leq N \int_{B(x_0,r)} |\tilde{p}_{x_0,\rho}|^{3/2} + |\tilde{p}_{x_0,r}|^{3/2} dx.
 \end{aligned}$$

From (3.6), (3.7), (3.3) and (3.4), we get

$$\begin{aligned}
 & \int_{B(x_0,r)} |p - [h_{x_0,r}]_{x_0,r}|^{3/2} dx \leq N \left(\int_{B(x_0,\rho)} |\nabla u|^2 dx \right)^{3/2} \\
 &+ N \left(\int_{B(x_0,\rho)} |f|^{6/5} dx \right)^{5/4} + N \left(\frac{r}{\rho} \right)^{15/2} \int_{B(x_0,\rho)} |p - [h_{x_0,\rho}]_{x_0,\rho}|^{3/2} dx.
 \end{aligned}$$

Finally, by Hölder's inequality, the lemma is proved. \square

Note that the following estimates use the equation (1.1), or more precisely they use the generalized local energy inequality (2.2).

Lemma 3.3. *Suppose $\theta \in (0, 1/2]$ and $\rho > 0$ are constants, and $B(x_0, \rho) \subset \Omega$. Then we have*

$$A(\theta\rho) + E(\theta\rho) \leq N\theta^{-2} [C^{2/3}(\rho) + C(\rho) + C^{1/3}(\rho)D^{2/3}(\rho) + F(\rho)].$$

In particular, when $\theta = 1/2$ we have

$$(3.8) \quad A(\rho/2) + E(\rho/2) \leq N[C^{2/3}(\rho) + C(\rho) + C^{1/3}(\rho)D^{2/3}(\rho) + F(\rho)].$$

Here N is a positive constant independent of θ , ρ and x_0 .

Proof. Let $r = \theta\rho$. By Hölder's inequality,

$$A(r) \leq C^{2/3}(r) \leq N\theta^{-2}C^{2/3}(\rho).$$

To estimate $E(r)$, in the energy inequality (2.2) we choose a suitable smooth cut-off function $\psi = \psi_1 \in C_0^\infty(B(x_0, \rho))$ such that

$$0 \leq \psi_1 \leq 1 \text{ in } B(x_0, \rho), \quad \psi_1 \equiv 1 \text{ in } B(x_0, \rho/2)$$

$$(3.9) \quad |\nabla \psi_1| \leq N\rho^{-1}, \quad |\nabla^2 \psi_1| \leq N\rho^{-2} \text{ in } B(x_0, \rho).$$

By using (2.2) and because u is divergence free, we get

$$\begin{aligned}
 E(r) &\leq \frac{N}{r^2} \left[\frac{1}{\rho^2} \int_{B(x_0,\rho)} |u|^2 dx \right. \\
 &\quad \left. + \frac{1}{\rho} \int_{B(x_0,\rho)} (|u|^2 + 2|p - [h]_{x_0,\rho}|)|u| dx + \int_{B(x_0,\rho)} |u||f| dx \right].
 \end{aligned}$$

Due to the Hölder inequality and Young's inequality, one obtains

$$\int_{B(x_0, \rho)} |u|^2 dx \leq \left(\int_{B(x_0, \rho)} |u|^3 dx \right)^{2/3} \left(\int_{B(x_0, \rho)} dx \right)^{1/3} \leq \rho^4 C^{2/3}(\rho).$$

And

$$\begin{aligned} \int_{B(x_0, \rho)} |p - [h]_{x_0, \rho}| |u| dx \\ \leq \left(\int_{B(x_0, \rho)} |p - [h]_{x_0, \rho}|^{3/2} dx \right)^{2/3} \left(\int_{B(x_0, \rho)} |u|^3 dx \right)^{1/3} \\ \leq N \rho^3 D^{2/3}(\rho) C^{1/3}(\rho). \end{aligned}$$

Furthermore

$$\int_{B(x_0, \rho)} |u| |f| dx \leq \frac{1}{\rho^2} \int_{B(x_0, \rho)} |u|^2 dx + \rho^2 \int_{B(x_0, \rho)} |f|^2 dx.$$

Then, collecting these estimates, Lemma 3.3 thus follows. \square

As a conclusion of this subsection, we obtain

Proposition 3.4. *For any small $\varepsilon_0 > 0$, there exists $\varepsilon_1 = \varepsilon_1(\varepsilon_0) > 0$ small such that for any $x_0 \in \Omega$ satisfying*

$$(3.10) \quad \limsup_{r \downarrow 0} E(r) \leq \varepsilon_1,$$

we have

$$(3.11) \quad A(\rho_0) + E(\rho_0) + C(\rho_0) + D(\rho_0) \leq \varepsilon_0,$$

provided that ρ_0 is sufficiently small.

Proof. For a given $x_0 \in \Omega$ satisfying (3.10), choose $\rho_1 > 0$ such that $B(x_0, \rho_1) \subset \Omega$. Then for any $\rho \in (0, \rho_1]$, by using (3.8) and Young's inequality

$$A(\gamma\rho) + E(\gamma\rho) \leq N[C^{2/3}(2\gamma\rho) + C(2\gamma\rho) + D(2\gamma\rho) + F(2\gamma\rho)].$$

This estimate, (3.1) and (3.2), with $\gamma \in (0, 1/6)$, together with Young's inequality again implies

$$\begin{aligned} (3.12) \quad & A(\gamma\rho) + E(\gamma\rho) + C(\gamma\rho) + D(\gamma\rho) \\ & \leq N [\gamma^2 C^{2/3}(\rho) + \gamma^{9/2} D(\rho) + \gamma^3 C(\rho) + \gamma^3 A(\rho)] \\ & \quad + N \gamma^{-50} [E(\rho) + E^3(\rho) + F(\rho)] + N \gamma^2 \\ & \leq N \gamma^2 [A(\rho) + E(\rho) + C(\rho) + D(\rho)] + N \gamma^2 \\ & \quad + N \gamma^{-50} [E(\rho) + E^3(\rho) + F(\rho)]. \end{aligned}$$

Since $f \in L_{6,\text{loc}}$, by Hölder's inequality, we have

$$(3.13) \quad F(\rho) \leq \|f\|_{L_6(B(x_0, \rho_1))}^2 \rho^4.$$

It is easy to see that for any $\varepsilon_0 > 0$, there are sufficiently small real numbers $\gamma \leq 1/\sqrt{2N}$ and ε_1 such that if (3.10) holds then for all small ρ we have

$$N\gamma^2 + N\gamma^{-50}(E(\rho) + E^3(\rho) + F(\rho)) < \varepsilon_0/2.$$

By using (3.12), we reach (3.11) for some $\rho_0 > 0$ small enough. \square

3.2. Step 2. In the second step, first we will estimate the values of A , E and D in a smaller ball by their values in a larger ball. Note that in this subsection all of the quantities implicitly depend upon the point x_1 as $A(r) = A(r, x_1)$ unless it says so otherwise.

Lemma 3.5. *Fix constants $\rho > 0$, $\theta \in (0, 1/3]$ and $B(x_1, \rho) \subset \Omega$. Then we have*

$$(3.14) \quad A(\theta\rho) + E(\theta\rho) \leq N\theta^2 A(\rho) \\ + N\theta^{-3}([A(\rho) + E(\rho)]^{3/2} + D(\rho)) + N\theta^{-6}F(\rho),$$

where $N > 0$ is independent of ρ , θ and x_1 .

Proof. We prove the lemma by using a suitably chosen test function in the generalized local energy inequality (2.2). Let $r = \theta\rho$. We define

$$\psi_2(x) = (r^2 + |x - x_1|^2)^{-2},$$

which clearly satisfies $\Delta\psi_2 = -24r^2(r^2 + |x - x_1|^2)^{-4}$ so that

$$(3.15) \quad \Delta\psi_2 < 0 \quad \text{in } \mathbb{R}^6, \quad \Delta\psi_2 \leq -cr^{-6} \quad \text{in } B(x_1, \rho),$$

for some constant $c > 0$ independent of r .

In the energy inequality (2.2) we choose $\psi = \psi_1\psi_2$, where ψ_1 is taken from (3.9) in the proof of Lemma 3.3 with the center x_1 in place of x_0 . Then we have

$$(3.16) \quad - \int_{B(x_1, \rho)} |u|^2 \psi_1 \Delta\psi_2 \, dx + 2 \int_{B(x_1, \rho)} |\nabla u|^2 \psi_1 \psi_2 \, dx \\ \leq \int_{B(x_1, \rho)} \{ |u|^2 (\psi_2 \Delta\psi_1 + 2\nabla\psi_1 \cdot \nabla\psi_2) \\ + (|u|^2 + 2(p - [h]_{x_1, \rho})) u \cdot (\psi_1 \nabla\psi_2 + \psi_2 \nabla\psi_1) + f \cdot u \psi_1 \psi_2 \} \, dx.$$

After some straightforward computations, from (3.9) and (3.15), it is easy to see the following properties:

(i) For some constant $c > 0$, on $\bar{B}(x_1, r)$ it holds that

$$\psi_1\psi_2 = \psi_2 \geq cr^{-4}, \quad -\psi_1\Delta\psi_2 = -\Delta\psi_2 \geq cr^{-6}.$$

(ii) In $B(x_1, \rho)$, we have

$$\begin{aligned} |\psi_1\psi_2| &\leq Nr^{-4}, \quad |\psi_1\nabla\psi_2| + |\psi_2\nabla\psi_1| \leq Nr^{-5}, \\ |\psi_2\Delta\psi_1| + |\nabla\psi_1 \cdot \nabla\psi_2| &\leq N\rho^{-6}. \end{aligned}$$

These properties together with (3.16), the Young and Hölder inequalities, yield

$$(3.17) \quad A(r) + E(r) \leq N[\theta^2 A(\rho) + \theta^{-3}(C(\rho) + D(\rho)) + \theta^{-6}F(\rho)].$$

Owing to the Sobolev inequality (2.3), one easily gets

$$(3.18) \quad C(\rho) \leq N[A(\rho) + E(\rho)]^{3/2}.$$

Upon combining (3.17) and (3.18), the lemma is proved. \square

Lemma 3.6. *Suppose $\rho > 0$ is constant and $B(x_1, \rho) \subset \Omega$. Then we can find a $\theta_1 \in (0, 1)$ small, where θ_1 does not depend upon ρ , such that*

$$(3.19) \quad \begin{aligned} A(\theta_1\rho) + E(\theta_1\rho) + D^{2/3}(\theta_1\rho) &\leq \frac{1}{4}[A(\rho) + E(\rho) + D^{2/3}(\rho)] \\ &\quad + N(\theta_1)[A(\rho) + E(\rho) + D^{2/3}(\rho)]^{3/2} + N(\theta_1)[F(\rho) + F^{1/2}(\rho)], \end{aligned}$$

where N is a constant independent of ρ and x_1 .

Proof. Due to (3.2) and (3.14), for any $\gamma, \theta \in (0, 1/3]$, we have

$$(3.20) \quad \begin{aligned} D^{2/3}(\gamma\theta\rho) &\leq N[\gamma^3 D^{2/3}(\theta\rho) + \gamma^{-2}F^{1/2}(\theta\rho) + \gamma^{-2}E(\theta\rho)] \\ &\leq N\gamma^3\theta^{-2}D^{2/3}(\rho) + N\gamma^{-2}F^{1/2}(\rho) + N\gamma^{-2}\theta^2 A(\rho) \\ &\quad + N\gamma^{-2}\theta^{-3}[A(\rho) + E(\rho) + D^{2/3}(\rho)]^{3/2} + N\gamma^{-2}\theta^{-6}F(\rho), \end{aligned}$$

and from (3.14) we have

$$(3.21) \quad \begin{aligned} A(\gamma\theta\rho) + E(\gamma\theta\rho) &\leq N(\gamma\theta)^2 A(\rho) \\ &\quad + N(\gamma\theta)^{-3}[A(\rho) + E(\rho) + D^{2/3}(\rho)]^{3/2} + N(\gamma\theta)^{-6}F(\rho). \end{aligned}$$

Now we choose and fix θ sufficiently small and $\gamma = \theta^{4/5}$ such that

$$N[\gamma^3\theta^{-2} + \gamma^{-2}\theta^2 + (\gamma\theta)^2] \leq N\theta^{2/5} \leq 1/8.$$

Upon adding (3.20) and (3.21), we obtain

$$\begin{aligned} A(\gamma\theta\rho) + E(\gamma\theta\rho) + D^{2/3}(\gamma\theta\rho) \\ \leq \frac{1}{4}[A(\rho) + D^{2/3}(\rho)] + N[A(\rho) + E(\rho) + D^{2/3}(\rho)]^{3/2} \\ \quad + N[F(\rho) + F^{1/2}(\rho)], \end{aligned}$$

where N depends only on θ and γ . After putting $\theta_1 = \gamma\theta$, the lemma is proved. \square

In the next proposition we will study the decay property of A , E , C and D as the radius ρ goes to zero.

Proposition 3.7. *There exists $\varepsilon_0 > 0$ satisfying the following property. Suppose that for some $x_0 \in \Omega$ and $\rho_0 \in (0, 1)$ satisfying $B(x_0, \rho_0) \subset \Omega$ we have*

$$(3.22) \quad C(\rho_0, x_0) + D(\rho_0, x_0) + F(\rho_0, x_0) \leq \varepsilon_0.$$

Then we can find $N > 0$ and $\alpha_0 \in (0, 1)$ such that for any $\rho \in (0, \rho_0/8)$ and $x_1 \in B(x_0, \rho_0/8)$, the following inequality will hold uniformly

$$(3.23) \quad A(\rho, x_1) + E(\rho, x_1) + C^{2/3}(\rho, x_1) + D^{2/3}(\rho, x_1) \leq N\rho^{\alpha_0},$$

where N is a positive constant independent of ρ and x_1 .

Proof. Fix the constant θ_1 from Lemma 3.6. Due to (3.8), (3.2) and (3.22), we may first choose $\varepsilon' > 0$ then $\varepsilon_0 = \varepsilon_0(\varepsilon') > 0$ sufficiently small such that

$$A(\rho_0/4, x_0) + E(\rho_0/4, x_0) + D^{2/3}(\rho_0/4, x_0) \leq \frac{\varepsilon'}{16},$$

and

$$(3.24) \quad N(\theta_1)\sqrt{\varepsilon'} \leq 1/4, \quad N(\theta_1)(\varepsilon_0 + \varepsilon_0^{1/2}) \leq \varepsilon'/2.$$

where $N(\theta_1) > 0$ is the same constant from (3.19). Since we suppose that $x_1 \in B(x_0, \rho_0/8)$, we then have

$$B(x_1, \rho_0/8) \subset B(x_0, \rho_0/4) \subset \Omega,$$

and then

$$\varphi(\rho_0) := A(\rho_0/8, x_1) + E(\rho_0/8, x_1) + D^{2/3}(\rho_0/8, x_1) \leq \varepsilon'.$$

By using (3.24) and (3.19) with $\rho = \rho_0/8$ we obtain inductively that

$$\varphi(\theta_1^k \rho_0) = A(\theta_1^k \rho_0/8, x_1) + E(\theta_1^k \rho_0/8, x_1) + D^{2/3}(\theta_1^k \rho_0/8, x_1) \leq \varepsilon'.$$

(Holding for $k = 1, 2, \dots$). It then similarly follows from (3.24) and (3.19) that

$$(3.25) \quad \varphi(\theta_1^k \rho_0) \leq \frac{1}{2}\varphi(\theta_1^{k-1} \rho_0) + N_1(\theta_1^{k-1} \rho_0)^2.$$

Above, thanks to (3.13), we have used the estimate

$$F(\theta_1^{k-1} \rho_0/8, x_1) + F^{1/2}(\theta_1^{k-1} \rho_0/8, x_1) \leq N_1(\|f\|_{L_6(B(x_0, \rho_0/2))}) (\theta_1^{k-1} \rho_0)^2.$$

Now we use a standard iteration argument to obtain the Hölder continuity of φ . We have to be a bit careful however because we do not

make the standard assumption that $\varphi(\rho)$ should be a non-decreasing function. We iterate (3.25) to obtain

$$\begin{aligned} \varphi(\theta_1^k \rho_0) &\leq \left(\frac{1}{2}\right)^k \varphi(\rho_0) + N_1 \rho_0^2 \sum_{j=0}^{k-1} \left(\frac{1}{2}\right)^j (\theta_1^{k-1-j})^2 \\ &\leq \left(\frac{1}{2}\right)^k \left[\varphi(\rho_0) + \frac{2N_1}{1-\theta_1} \rho_0^2 \right]. \end{aligned}$$

In the last inequality, without loss of generality we have used that $\theta_1 \in (0, 1/2]$. Since $\rho \in (0, \rho_0/8)$ we can find k such that $\theta_1^k \frac{\rho_0}{8} < \rho \leq \theta_1^{k-1} \frac{\rho_0}{8}$. Then

$$\begin{aligned} A(\rho, x_1) + E(\rho, x_1) + D^{2/3}(\rho, x_1) &\leq \theta_1^{-4} \varphi(\theta_1^{k-1} \rho_0) \\ &\leq 2\theta_1^{-4} \left(\frac{1}{2}\right)^k \left[\varphi(\rho_0) + \frac{2N_1}{1-\theta_1} \rho_0^2 \right] \leq N \left(\frac{\rho}{\rho_0}\right)^{\alpha_0}. \end{aligned}$$

In this last line $N = N(\theta_1, \varphi(\rho_0), N_1, \rho_0)$ and $\alpha_0 = \frac{\log(1/2)}{\log(\theta_1)} > 0$. This yields (3.23) for the terms A , E and D . The inequality for $C(\rho, x_1)$ follows from (3.18). \square

3.3. Step 3 – Proofs of Theorems 2.1-2.3. In the final step, we are going to use a bootstrap argument to successively improve the decay estimate (3.23). However, as we will show below, the bootstrap argument itself only gives the decay of $E(\rho)$ no more than ρ^2 , i.e. one can obtain an estimate like

$$\int_{B(x_1, \rho)} |\nabla u|^2 dx \leq N(\varepsilon) \rho^{4-\varepsilon}, \quad \forall \varepsilon > 0,$$

for any ρ sufficiently small. Unfortunately, this decay estimate is not enough for the Hölder regularity of u since the dimension is six (so that we need the exponent $4 + \varepsilon$ instead of $4 - \varepsilon$ according to the Morrey lemma). Then to fill in this gap we will use the elliptic theory.

First we prove Theorem 2.2. We begin with the bootstrap argument. We will choose an increasing sequence of real numbers $\{\alpha_k\}_{k=1}^\infty \in (\alpha_0, 2)$ such that for any small $\delta > 0$ we can find an integer $m = m(\delta)$ with the property that $\alpha_m > 2 - \delta$.

For a fixed $\delta > 0$ and $m = m(\delta)$, under the condition (2.5), we *claim* that the following estimates hold uniformly for all $\rho > 0$ sufficiently small and $x_1 \in B(x_0, \rho_0/8)$ over the range of $\{\alpha_k\}_{k=0}^m$:

$$(3.26) \quad \begin{aligned} A(\rho, x_1) + E(\rho, x_1) &\leq N \rho^{\alpha_k}, \\ C(\rho, x_1) &\leq N \rho^{3\alpha_k/2}, \quad D(\rho, x_1) \leq N \rho^{3\alpha_k/2}. \end{aligned}$$

We prove this via iteration. The $k = 0$ case for (3.26) with α_0 was proven in (3.23).

We first estimate $A(\rho, x_1)$ and $E(\rho, x_1)$. Let $\rho = \tilde{\theta}\tilde{\rho}$ where $\tilde{\theta} = \rho^\mu$, $\tilde{\rho} = \rho^{1-\mu}$ and $\mu \in (0, 1)$ to be determined. We use Lemma 3.5 and then (3.26) (for α_k) to obtain

$$A(\rho) + E(\rho) \leq N \left(\rho^{2\mu + \alpha_k(1-\mu)} + \rho^{\frac{3}{2}\alpha_k(1-\mu) - 3\mu} + \rho^{4(1-\mu) - 6\mu} \right).$$

Choose $\mu = \frac{\alpha_k}{10 + \alpha_k}$, then (3.26) is proven for $A(\rho) + E(\rho)$ with the exponent of

$$\alpha_{k+1} = \frac{12}{10 + \alpha_k} \alpha_k \in (\alpha_k, 2).$$

Then the estimate in (3.26) (with α_{k+1}) for $C(\rho, x_1)$ follows from (3.18). To prove the estimate in (3.26) (after level k) for $D(\rho, x_1)$ we will use Lemma 3.2. From (3.2) we obtain

$$D(\gamma\rho, x_1) \leq N \left(\gamma^{9/2} D(\rho, x_1) + \gamma^{-3} \rho^{3\alpha_{k+1}/2} + \gamma^{-3} \rho^3 \right).$$

The estimate used here for $F(\rho)$ follows from (3.13). Now for any r small, we take the supremum on both sides with respect to $\rho \in (0, r)$ and get

$$\sup_{\rho \in (0, r]} D(\gamma\rho, x_1) \leq N \gamma^{9/2} \sup_{\rho \in (0, r]} D(\rho, x_1) + N \gamma^{-3} r^{3\alpha_{k+1}/2} + N \gamma^{-3} r^3.$$

Since $9/2 > 3 > \frac{3}{2}\alpha_{k+1}$, by using a well-known iteration argument, similar to (3.25) (or see e.g., [12, Chap. 3, Lemma 2.1]), we obtain the estimate in (3.26) (with α_{k+1}) for $D(\rho)$. Then we have shown how to build the increasing sequence of $\{\alpha_k\}$ for which (3.26) holds. Moreover,

$$2 - \alpha_{k+1} = \frac{10}{10 + \alpha_k} (2 - \alpha_k) \leq \frac{10}{10 + \alpha_0} (2 - \alpha_k),$$

which implies that $\alpha_k \rightarrow 2$ as $k \rightarrow \infty$. Note that by the above proof, the constant N in (3.26) may go to infinity as $k \rightarrow \infty$; thus we truncate at level $m < \infty$.

In particular, (3.26) with $k = m$ gives for any small $\delta = \delta(m) > 0$ that

$$(3.27) \quad \int_{B(x_1, \rho)} |u|^2 dx \leq N \rho^{6-\delta},$$

$$(3.28) \quad \int_{B(x_1, \rho)} |u|^3 + |p - [h]_{x_1, \rho}|^{3/2} dx \leq N \rho^{6-\frac{3}{2}\delta}.$$

We obtained these estimates via the bootstrap argument, next we will use the elliptic theory to improve them.

Now we fix a $\delta \in (0, 1/10)$ and rewrite (1.1) (in the sense of distributions) into

$$\Delta u_i = D_j(u_i u_j) + D_i p - f_i.$$

Finally, we use the classical elliptic theory to complete the proof. Thanks to (3.27), there exists $\rho_1 \in (\rho/2, \rho)$ such that

$$(3.29) \quad \int_{S(x_1, \rho_1)} |u|^2 dx \leq N \rho^{5-\delta}.$$

Let v be the unique H^1 solution to the Laplace equation

$$\Delta v_i = 0 \quad \text{in } B(x_1, \rho_1),$$

with the boundary condition $v_i = u_i$ on $S(x_1, \rho_1)$. It follows from the standard estimates for harmonic functions, Hölder's inequality, and (3.29) that

$$(3.30) \quad \sup_{B(x_1, \rho_1/2)} |\nabla v| \leq N \rho_1^{-6} \int_{S(x_1, \rho_1)} |v| dx \leq N \rho^{-1-\delta/2}.$$

Denote $w = u - v \in H^1(B(x_1, \rho_1))$. Then w satisfies the Poisson equation

$$\Delta w_i = D_j(u_i u_j) + D_i(p - [h]_{x_1, \rho}) - f_i \quad \text{in } B(x_1, \rho_1).$$

with zero boundary condition on $S(x_1, \rho_1)$. By the classical L_p estimates for the Poisson equation, we have

$$\begin{aligned} \|\nabla w\|_{L_{3/2}(B(x_1, \rho_1))} &\leq N \| |u|^2 \|_{L_{3/2}(B(x_1, \rho_1))} \\ &\quad + N \|p - [h]_{x_1, \rho}\|_{L_{3/2}(B(x_1, \rho_1))} + N \rho_1 \|f\|_{L_{3/2}(B(x_1, \rho_1))}. \end{aligned}$$

This together with the assumption on f and (3.28) gives

$$(3.31) \quad \|\nabla w\|_{L_{3/2}(B(x_1, \rho_1))} \leq N \rho^{4-\delta} + N \rho^4 \leq N \rho^{4-\delta}.$$

Since $|\nabla u| \leq |\nabla w| + |\nabla v|$, we combine (3.30) and (3.31) to obtain, for any $r \in (0, \rho/4)$, that

$$\int_{B(x_1, r)} |\nabla u|^{3/2} dx \leq N \rho^{6-3\delta/2} + N r^6 \rho^{-3/2-3\delta/4}.$$

Upon taking $r = \rho^{5/4-\delta/8}/4$ (with ρ small), we get

$$(3.32) \quad \int_{B(x_1, r)} |\nabla u|^{3/2} dx \leq N r^\beta,$$

where

$$\beta = \frac{6 - 3\delta/2}{5/4 - \delta/8} > 6 - 3/2.$$

Since (3.32) holds for arbitrary $x_1 \in B(x_0, \rho_0/8)$ and all r small, by the Morrey lemma (see for instance [12, Theorem 1.1 on p. 64 of Ch. III]), u is Hölder continuous in a neighborhood of x_0 . This completes the proof of Theorem 2.2.

Theorem 2.1 then follows from Theorem 2.2 by applying Proposition 3.4. Finally, Theorem 2.3 is deduced from Theorem 2.1 by using the standard argument in the geometric measure theory, which is explained for example in [1], or alternatively in [12].

Remark 3.8. Finally we remark that by using the same method we can get an alternative proof of Theorem 1.1 for the 5D steady-state Navier-Stokes equations if we assume that $f \in L_{5,\text{loc}}$. However, it seems to us that six is the highest dimension to which our approach (or any existing approach) applies. In fact, by the Sobolev imbedding theorem, $L_3(\mathbb{R}^6) \hookrightarrow H^1(\mathbb{R}^6)$. So the nonlinear term in the energy inequality can be controlled by the energy norm when $d = 6$ but not higher.

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