Algebraic Topology (my minor)

1. If M^3 is a compact threefold with $\pi_1 M = 0$, then M^3 is homotopic to S^3.
2. Give examples of spaces with isomorphic homotopy and nonisomorphic homology and vice versa.
3. Let X have $H^*(X)=\mathbb{Z}[\alpha]$ with $|\alpha| = 2$. Show that it is homotopic to $K(\mathbb{Z}, 2)$.
4. Show that $\mathbb{RP}^2$ cannot be embedded in $\mathbb{R}^3$.
5. Show that $TS^6 \to S^6$ has no proper subbundles.
6. Compute the cohomology of $K(\mathbb{Z}, 3)$ through dimension 8.
7. Compute $H^*(\Omega^2 S^n, \mathbb{Q})$.
8. Compute the Stiefel-Whitney classes for $\mathbb{RP}^3$ through $\mathbb{RP}^9$ and find the smallest k such that the Stiefel-Whitney classes dont obstruct embedding $\mathbb{RP}^n \to \mathbb{R}^k$.
9. Define the Stiefel-Whitney numbers of a manifold by taking any sequence $(i_1, \ldots, i_\ell)$ such that $i_1 + \ldots + i_\ell = \dim M$, and evaluating $w_{i_1}(TM) \ldots w_{i_\ell}(TM)$ on the fundamental class of the manifold (with $\mathbb{Z}/2$ coefficients). Show that if M is the boundary of a manifold W, then all the Stiefel-Whitney classes are zero. Which of the small $\mathbb{RP}^n$s are not boundaries? For the ones that may be, attempt to find the bounding manifold.

Algebraic Geometry (first advisor for orals)

1. Let X be the blowup of $\mathbb{P}^2$ at nine points. Compute the canonical bundle, the Hodge numbers, the number of complex moduli. When does there exist a map $\pi : X \to \mathbb{P}^1$ with elliptic fibers? Compute the Picard group. If $\mathscr{L}$ is a line bundle on $\mathbb{P}^1$, and $i : \mathbb{P}^1 \to X$ is the inclusion of an exceptional divisor, then use the Leray Spectral Sequence to compute $R^j \pi_*(i_* \mathscr{L})$ when $\pi : X \to \mathbb{P}^1$ is an elliptic fibration. Let $\mathscr{L}$ be a line bundle on X. Compute $H^i(X, \mathscr{L})$.
2. Look at the curve $y^2 = \prod_{i=1}^{5} (x - i)^i$. What can be said about the geometry? Are there any special g^r_ds? Can you write it, locally around $(0 : 1 : 0)$ as a manifestly hyperelliptic curve? (HINT: use as one coordinate $\frac{1}{x}$.)
3. Let C be a curve of genus 7, and let $J = Jac(C)$ the Jacobian. There is a natural divisor $\Theta \in \mathrm{Pic}(J)$. Describe Θ as a variety. (ie, singularities, irreducibility, etc). Work out the stratification on the singular locus (ie, the dimension of the locus at each multiplicity), write out explicitly a plane model of C as a sextic with three nodes. (Can be done because g^2_6 exist and are singular points of Θ.)

4. Embed the Del Pezzo surface obtained by blowing up 4 or 5 points in $\mathbb{P}^2$ via $-K_X$. How many lines are there? How many pencils of conics? What are the orders of the automorphism groups of the configuration of lines and of the surfaces?

5. On the fifth Del Pezzo surface, each point defines two pencils of conics, one of lines through it and one of conics through the other four. How are they related?

6. Work out the moduli space of DP_5s.

7. A curve of genus 6 is known to have 5 g^1_4s. Connect that to DP_4s. Hint: on a curve of genus 6, $K - D$ for D a g^1_4 will be a g^2_6 which maps the curve as a sextic with four singular points. What happens when you blow them up? Describe what the Cremona transformations do to the g^1_4s.

8. Take five lines in the plane intersecting as a pentagon. Choose two of them. Check that there exists a double cover of the plane branched over the union of these two lines. Now fix one line and let the other vary, this gives four pairs. Take the fiber product of these double covers. This gives a cover of degree 16. What happens if you choose a different line? What is the Galois group of this cover? Can you identify the surface abstractly?

9. Prove that d points on a rational normal curve are linearly independent or else span the entire projective space.

10. Prove that on a hyperelliptic curve, a g^r_d can be written as $rg^1_2 + E$ where E is the base locus. Do this in two ways.

11. Prove that every curve has a g^1_{g+1}. How many are there? Is $g + 1$ the lowest d such that every curve of genus g has a g^1_d? Pick some d with the property that every curve of genus g has one. Then look at the space of all genus g and degree d covers of $\mathbb{P}^1$. Show that this space is irreducible. Use this to show that the coarse moduli space of curves $\mathcal{M}_g$ is irreducible. Hint: Use monodromy.

Algebraic Geometry (second advisor for orals)

1. Check that the following methods prove the existence of smooth curves of arbitrary genus in the quadric surface, with divisor class $(g + 1, 2)$:
   (a) Given $\mathbb{P}^1$ with $2g + 2$ distinct points, describe the set of smooth double covers branched at those points. Can any be embedded in the quadric surface?
   (b) Note that you can describe curves in $\mathbb{P}^1 \times \mathbb{P}^1$ by having them homogeneous of bidegree $(g + 1, 2)$. Check that there is a smooth curve in this way.
   (c) Use Bertinis Theorem to show that a generic divisor in the linear system $(g + 1, 2)$ is smooth. Check dimension and base points.

2. Are there always nonhyperelliptic curves of any genus in the quadric surface?

3. Note that Grothendieck Riemann-Roch says that for $f : X \to Y$ a map of smooth varieties and $\mathscr{F}$ a sheaf on X, we have

$$f_*(ch(\mathscr{F})td(X)) = ch(Rf_*(\mathscr{F}))td(Y)$$

in $H^*(Y)$. Use this to compute $\pi_*(\mathscr{O}_C)$ where $C \to \mathbb{P}^1$ is the g^1_2 on a smooth hyperelliptic curve of genus g.

4. Let C be a smooth hyperelliptic curve. Does it have a g^1_3?

5. Look at the linear system of curves of degree d in $\mathbb{P}^2$. Prove that the locus of singular curves is a divisor. Compute its degree.

6. Same as 5, but for $\mathbb{P}^1 \times \mathbb{P}^1$.

7. Compute the canonical classes of the quartic surface in $\mathbb{P}^3$, the transverse intersection of two quadrics in $\mathbb{P}^3$, and the transverse intersection of three quadrics in $\mathbb{P}^5$.

8. Can a smooth quartic surface contain a line? If so, find an example. If it contains a line, then the line is a divisor in S. Compute the cohomology. Does every quartic surface contain a line?

9. Take the quadric surface in $\mathbb{P}^3$. Take a curve of bidegree $(1, 4)$ and on of bidegree $(1, 0)$, both smooth. There exists a unique double cover branched at their union. It will be singular, so take a resolution. You get a smooth surface S. Project this onto the second $\mathbb{P}^1$. Describe the fibers: what is the genus of the general fiber? How many and what kinds of singular fibers occur, for general curves? Is S rational?