Text: Stein-Shakarchi: Princeton Lecture Notes in Analysis "Measure Theory, Integration, and Hilbert Spaces"

References: Real and Complex Analysis by Rudin, Dunford and Schwartz "Linear Operators Vol I"

Topics: Lebesgue Measure and Integration, $L^1(\mathbb{R}^n)$, Fundamental Theorem of Calculus/Lebesgue Theorem, $L^2(\mathbb{R}^n)$, Hilbert Spaces, L^p spaces, abstract Banach spaces

So why is the Riemann Integral not good enough? It is not well behaved with respect to pointwise limits.

With the Riemann integral, we can have a sequence of functions f_n that converge almost everywhere to f with $\int |f_n|^2(x)dx \leq 1$ and have f not be integrable, even if the f_n are smooth.

Example 0.1 (Bad Behavior of Riemann Integral). *A function $f(x) : [a,b] \to \mathbb{R}$ is Riemann integrable if there exists 2 sequences of increasing/decreasing step functions $\varphi_1(x) \leq \varphi_2(x) \leq \ldots \leq f(x) \leq \ldots \leq \psi_n(x) \leq \psi_{n-1}(x) \leq \ldots \leq \psi_1(x)$ such that for some $M \in \mathbb{R}$, $\sum_{x\in[a,b]} |\varphi_j(x)| \leq m$, $\sup_{x\in[a,b]} |\psi_j(x)| \leq M$ for all j and moreover, $\lim_{k\to\infty} \int_a^b \varphi_k(x)dx = \lim_{k\to\infty} \int_a^b \psi_k(x)dx$. If two such families exist, then we get the same limit for any other pair of such families by passing to a common refinement.*

Now we will construct a sequence of nice (in fact, smooth) functions which is decreasing, converges pointwise, but whose limit is not Riemann integrable. First we construct an auxiliary sequence of functions $F_k(x)$, $k = 1, \ldots$ as follows. Define $F_1(x)$ to be the function from $[0,1] \to \mathbb{R}$ such that for x not in a ball of diameter c_1, $F_1(x) = 1$ and that decreases to 0 and back to 1 linearly on the ball.

Define $F_2(x)$ such that around $1/4, 3/4$ we have balls of diameter c_2 where $2c_2 + c_1 < 1$ and the three balls do not overlap such that on each c_2 diameter ball, F_2 looks like F_1, and is constant in the middle. Continue inductively like this. By smoothing out the kinks, we can achieve that each $F_i(x)$ is C^∞ and $0 \leq F_i(x) \leq 1$ for all i.

We ensure that, upon putting $|c_k^i| = \ell_k$, $1 \leq i \leq k-1$, $\sum_{k=1}^\infty 2^{k-1}\ell_k < 1$. Then $f_n(x) = \prod_{k=1}^n F_k(x)$ is a decreasing sequence of bounded functions, so $\lim_{n\to\infty} f_n(X) = f(x)$ exists for all $x \in [0,1]$.

Letting $\hat{C} = \cup_{k=1}^\infty \cup_{j=1}^{2^{k-1}} C_k^i$, then $f(x) = 1$ on $I \setminus \hat{C}$ and furthermore it is discontinuous everywhere on $I \setminus \hat{C}$.

Claim: f is not Riemann Integrable. This is due to the following theorem:

Theorem 0.1. *If a function $f : I \to \mathbb{R}$ is Riemann Integrable, then its set of discontinuities has measure 0.*

Proof. Let $f : I \to \mathbb{R}$ be Riemann Integrable, in particular, bounded. For $c \in I$, r sufficiently small, define $osc(f,c,r) = \sup_{x,y\in I_r(c)} |f(x) - f(y)|$ where $I_r(c)$ is the interval of length r centered at c. Further, define $osc(f,c) = \lim_{r\to 0} osc(f,c,r)$. Then the set of discontinuities of f is $\{x \in I | osc(f,x) > 0\}$.

Denoting $A_\epsilon = \{x \in I | osc(f,x) \geq \epsilon\}$, then note that A_ϵ is closed, and the set of discontinuities of f is $\cup_{n=1}^{\infty} A_{1/n}$. We will show that each $A_{1/n}$ satisfies $|A_{1/n}| = 0$, so $|\cup A_{1/n}| = 0$.

Pick a set $A_{1/n}$. Further, given $\epsilon > 0$, pick a lower and upper bounding step function $\varphi(x) \leq f(x) \leq \psi(x)$ subordinate [$\varphi(x), \psi(x)$ are constant on I_k for all k] to the cover $I = \cup_{k=1}^{N} I_k$ and $0 \leq \int_I \psi(x)dx - \int_I \varphi(x)dx < \epsilon/n$ implies $S = \sum_{I_k^\circ \cap A_{1/n} \neq \emptyset} |I_k| < \epsilon$. This is because $\epsilon/k \geq \int_I (\psi(x) - \varphi(x))dx \geq \sum_{I_k^\circ \cap A_{1/n} \neq \emptyset} |I_n| \frac{1}{n} = \frac{1}{n} S$ □

1 Measure Theory: Lebesgue Measure on $\mathbb{R}^n$

First, we will define Lebesgue measure on rectangles:

Definition 1.1 (Rectangle, Measure of a Rectangle)**.** *A closed rectangle on $\mathbb{R}^n$ is a set of the form $[a_1, b_1] \times [a_2, b_2] \times \ldots [a_n, b_n]$ with $a_i < b_i$ for all i.*

We define $|R| = \prod_{i=1}^{n} (b_i - a_i)$.

This definition is compatible with decomposition of rectangles into smaller rectangles.

Lemma 1.1. *Let $R = \cup_{j=1}^{M} R_j$ be an almost disjoint cover. This means that only the boundaries may intersect. Then $|R| = \sum_{j=1}^{M} |R_j|$.*

Proof. Refine the rectangles R_j by introducing a suitable $\tilde{R}_j$ such that $R = \cup_{j=1}^{N} \tilde{R}_j$ and each $R_j = \cup_{j \in \Lambda_k} \tilde{R}_j$, again with the $\tilde{R}_j$ almost disjoint. Now the statemeny $|R| = \sum |\tilde{R}_j|$ follows from the distribution law for the reals, and similarly $|R_k| = \sum_{j \in \Lambda_k} |\tilde{R}_j|$ follows from distribution. As teh Λ_k partition the set $\{1, \ldots, N\}$, we have that $|R| = \sum_{k=1}^{M} |R_k|$. □

Lemma 1.2. *If $R \subset \cup R_j$, then $|R| \leq \sum_{j=1}^{M} |R_j|$.*

Proof. Basically the same, but the index sets Λ_k are no longer necessarily disjoint. □

Theorem 1.3. *Every open set $O \subset \mathbb{R}^n$ can be written as a union of almost disjoint closed cubes.*

Proof. For each point $x \in O$, pick the largest dyadic cube (cube on $2^k \mathbb{Z}^n$, $k \in \mathbb{Z}$) still in O containing x. This gives the disjoint cubes. □

This uses the fact that if you have two dyadic cubes, Q_1, Q_2, then if $Q_1^\circ \cap Q_2^\circ \neq \emptyset$, then either $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$.

Now we need to define measurable sets.

Definition 1.2 (Outer Measure)**.** *Let $E \subset \mathbb{R}^n$ be any subset, then we define $m_*(E) = \inf_{E \subset \cup_{j=1}^{\infty} Q_j} |Q_j|$ where the Q_j are cubes. In particular, $m_*(E) \in [0, \infty]$.*

It is important to allow countable unions here. If one restricts to only finitely many cubes, this is claled the outer Jordan content, $j_*(E) \neq m_*(E)$. For instance, $I \cap \mathbb{Q}$ has $j_*(I \cap \mathbb{Q}) = 1$ and $m_*(I \cap \mathbb{Q}) = 0$.

Some facts concerning m_*:

1. $m_*(pt) = 0$
2. For a rectangle R, $m_*(R) = |R|$

Proof of (2): Assume $R \subset \cup_{j=1}^{\infty} Q_j$. Choose open cubes $\tilde{Q}_j^\circ \supset Q_j$. By compactness of R, we can choose a finite collection of $\tilde{Q}_j^\circ$ which still covers R. Then by the lemmas, $\sum |\tilde{Q}_j^\circ| \geq |R|$. Let $\epsilon > 0$, then assume that $\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \epsilon$, and also $|\tilde{Q}_j^\circ| \leq (1+\epsilon)|Q_j|$.

Then $(1+\epsilon)(\epsilon + m_*(R)) \geq |R|$, so now letting $\epsilon \to 0$, get $m_*(R) \geq |R|$. Next we prove the reverse. Construct a grid of cubes of side length $1/k$ for k large and let $\{Q_j\}_{j \in \Lambda}$ be those (finitely many) cubes intersecting R. Then subdivide Λ into $A \cup B$, where $A = \{j \in \Lambda | Q_j \subset R\}$ and $B = \{j \in \Lambda | Q_j \not\subset R\}$. Then check directly that there is a constant $c = c(d, R)$ such that $\#B \leq ck^{d-1}$, if $R \subset \mathbb{R}^d$.

Further, $\sum_{j \in A} |Q_j| \leq |R|$ by lemma 2. Hence, $|R| + ck^{-1} \geq \sum_{j \in \Lambda} |Q_j|$, so now letting $k \to \infty$, we get $m_*(R) \leq |R|$. So $|R| = m_*(R)$.

Remark 1.1. *This allows us to replace cubes by rectangles in the definition of* m_*

2 Lecture 2

Last time, we constructed the outer measure for any set $E \subset \mathbb{R}^n$.

The following are fundamental properties of $m_*(E)$:

1. If R is a rectangle, then $m_*(R) = |R|$ (showed last time)
2. Monotonicity: if $E_1 \subset E_2$ then $m_*(E_1) \leq m_*(E_2)$
3. Countable sub-additivity: If $E = \cup_{i=1}^{\infty} E_i$ then $m_*(E) \leq \sum m_*(E_i)$
4. Approximation by open sets: $m_*(E) = \inf_{O \supset E \text{open}} m_*(O)$.

   Proof. Given a covering $E \subset \cup Q_j$ by closed cubes, choose a small open thickening of each. □

5. Additivity of m_* for well separated sets: We call E_1, E_2 well-separated proved that $d(E_1, E_2) = \inf_{x \in E_1, y \in E_2} |x - y| > 0$. Then if E_1, E_2 are well-separated, then $m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$.

   Remark: It is not enough that E_1 and E_2 be disjoint for this to hole.

Proof. By sub additivity, we have $m_*(E_1 \cup E_2) \leq m_*(E_1) + m_*(E_2)$. So we need the other direction.

Choose a covering by cubes $\{Q_j\}_{j=1}^{\infty}$ such that $\sum |Q_j| < m_*(E_1 \cup E_2) + \epsilon$ by subdividing these cubes into smaller ones, we can ensure that they have sidelength at most $\frac{1}{2}d(E_1, E_2) > 0$, so no cube intersects both sets. Then, we can partition $\{Q_j\}$ into $\{Q_j\}_{j\in A}$ and $\{Q_j\}_{j\in B}$ where $j \in A$ iff $Q_j \cap E_1 \neq \emptyset$ and $j \in B$ else. But necessarily, $E_1 \subset \cup_{j\in A} Q_j$ and $E_2 \subset \cup_{j\in B} Q_j$, and so $m_*(E_1) \leq \sum_{j\in A} |Q_j|$ and $m_*(E_2) \leq \sum_{j\in B} |Q_j|$ and so $m_*(E_1) + m_*(E_2) \leq \sum |Q_j| \leq m_*(E_1, E_2) + \epsilon$. Taking $\epsilon \to 0$, we get $m_*(E_1) + m_*(E_2) \leq m_*(E_1 \cup E_2)$, and so equality holds. ☐

6. Countable Additivity for well-behaved decomposition: If $E = \cup_{j=1}^{\infty} Q_j$ and Q_j are almost disjoint, then $m_*(E) = \sum |Q_j|$

   Proof. This is a consequence of the preceding properties: we're trying to reduce this assertion to additivity of m_* on well-separated sets. Shrink each cube a bit. That is, we replace it by smaller, concentric cubes $\tilde{Q}_j \subset Q_j$, but such that $d(\tilde{Q}_j, \tilde{Q}_k) > 0$ for $i \neq k$. Then for any finite collection, applying the preceding property inductively gives us that $m_*(\cup_{j=1}^{M} \tilde{Q}_j) = \sum_{i=1}^{M} |\tilde{Q}_j|$. By letting the smaller cubes approach the original cubes, we conclude that $m_*(E) \geq \sum_{j=1}^{M} |Q_j|$ for every finite M. Now, letting $M \to \infty$, $m_*(E) \geq \sum |Q_j$, and the other direction is provided by sub-additivity. ☐

The problem of m_* is that it fails additivity for disjoint sets E_1, E_2. We'll restrict E to a class of sets in whcih m_* is countably additive.

Definition 2.1 (Lebesgue Measurable)**.** *We call a set $E \subset \mathbb{R}^n$ Lebesgue measurable proveded that $\forall \epsilon > 0$, there exists open $O \supset E$ with the property that $m_*(O \setminus E) < \epsilon$.*

In particular, open sets are Lebesgue measurable and sets of outer measure 0 are Lebesgue measurable.

Remark 2.1. *Alternatively, can characterize measurable sets as follows: for all $\epsilon > 0$, there exists closed $C \subset E$ such that $m_*(E \setminus C) < \epsilon$. This will be shown by checking that measurability is preserved under taking complements.*

Our goal now is to establish that the measurable sets in $\mathbb{R}^n$ constiture a σ-algebra.

Proposition 2.1. *A countable union of measurable sets is measurable.*

Proof. Choose $\epsilon > 0$ and for each measurable set E_j choose an open set O_j such that $m_*(O_j \setminus E_j) < \epsilon/2^j$.

Then, $O = \cup O_j$ satisfies $m_*(O \setminus \cup E_j) \leq \sum m_*(O_j \setminus E_j) < \epsilon$ by countable additivity. ☐

The hard part is showing that complementation preserves measurability.

Proposition 2.2. *Closed sets are measurable.*

Proof. Let $F \subset \mathbb{R}^n$ closed. Intersect F with balls $B_k(0)$ with radius $k \geq 1$. Then $F = \cup_{k=1}^{\infty}(F \cap B_k(0))$, each of which is compact. It we have measurability for $F \cap B_k$ for all k, then by the previous prop, we have measurability of F. Fix $k \geq 1$. Choose an open set O such that $m_*(O) \leq m_*(F \cap B_k) + \epsilon$ where $\epsilon > 0$ is fixed.

We want to show that $m_*(O \setminus (F \cap B_k)) < \epsilon$. Note that $O \setminus (F \cap B_k)$ is open. Hence, we can write it as a union of almost disjoint dyadic cubes. Also, be an earlier obervation, $m_*(O \setminus (F \cap B_k)) = \sum |Q_j|$, the dyadic cubes. Now pick a finite subset of these cubes $Q_1, \ldots, Q_M$. Then $\cup_{j=1}^{M} Q_j$ is a compact set, and so is $F \cap B_k$, hence $d(\cup_{j=1}^{M} Q_j, F \cap B_k) > 0$. And so by additivity for well-separated sets, we have $m_*(O) \geq m_*(\cup_{j=1}^{M} Q_j \cup F \cap B_k) = m_*(\cup_{j=1}^{M} Q_j) + m_*(F \cap B_k) = \sum_{j=1}^{M} |Q_j| + m_*(F \cap B_k)$, and so subtracting $m_*(F \cap B_k)$ from each side and recalling that $m_*(O) - m_*(F \cap B_k) < \epsilon$, we get $\sum_{j=1}^{M} |Q_j| M < \epsilon$, and now letting $M \to \infty$, $\sum_{j=1}^{\infty} |Q_j| \leq \epsilon$, and so $m_*(O \setminus F \cap B_k) \leq \epsilon$, so we get measurability of $F \cap B_k$ for all $k \geq 1$, and so F is measurable. □

Proposition 2.3. *Measurability is preserved under taking complements.*

Proof. Let $E \subset \mathbb{R}^n$ be measurable. For all $n \geq 1$ integer, choose $O_n \supset E$ such that $m_*(O_n \setminus E) < 1/n$, and $S = \cup O_n^c \subset E^c$, which is measurable. Further, $E^c \setminus S \subset O_n \setminus E$ for all $n \geq 1$. Hense, by monotonicity, $m_*(E^* \setminus S) < 1/n$ for all n, and so $m_*(E^c \setminus S) = 0$. Adn $E^c = E^c \setminus S \cup S$ is measurable, and so E^c is measurable. □

Corollary 2.4. *Countable intersections of measurable sets are measurable.*

Proposition 2.5 (Countable Additivity)**.** *Let $E_1, \ldots, \ldots$ be measurable and disjoint, then $m(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} m(E_j)$.*

Proof. Again uses compactness trick. First, reduce to bounded measurable sets by introducing $E_{j,k} = E_j \cap Q_k \setminus Q_{k-1}$. It is easy to see that countable additivity for $E_{j,k}$ implies the result for E_i, hence, we may assume that each E_i is bounded.

By measurability, for all j, there exists $F_j \subset E_j$ with F_j closed such that $m_*(E_j \setminus F_j) < \epsilon/2^j$ for a given $\epsilon > 0$. Now the F_j are disjoint and compact, hence, $d(F_i, F_j) > 0$ for $i \neq j$ and by the additivity of m_* on well-separated sets, we have $m_*(\cup_{j=1}^{M} F_j) = \sum_{j=1}^{M} m_*(F_j) \geq \sum_{j=1}^{M}(m(E_j) - \frac{\epsilon}{2^j})$, so $m(E) \geq \sum_{j=1}^{\infty} m(E_j)$ as $M \to \infty$ and $\epsilon \to 0$, and also $m(E) \leq \sum m(E_j)$ by countable subadditivity. □

Further consequences in the same vein:

Proposition 2.6. *Let $E_1, \ldots$ be an increasing sequence of measurable sets, that is, $E_k \subset E_{k+1}$ for all k, and $E = \cup E_k$, then one writes $E_k \nearrow E$, and similarly if $E_{k+1} \subset E_k$ and $E = \cap E_k$, then $E_k \searrow E$. Then we have:*

1. *If $E_k \nearrow E$, then $m(E) = \lim_{N\to\infty} m(E_N)$*

2. *If $E_k \searrow E$, then $m(E) = \lim_{N\to\infty} m(E_N)$ provided that $m(E_k) < \infty$ for some k.*

Proof. 1. Put $G_k = E_k \setminus E_{k-1}$, and $G_1 = E_1$. These are measurable and disjoint, and $E = \cup_{k=1}^{\infty} G_k$, and so

$$m(E) = \sum_{k=1}^{\infty} m(G_k) = \lim_{N\to\infty} \sum_{k=1}^{N} m(G_k) = \lim_{N\to\infty} m(E_N)$$

2. same idea.

□

Lemma 2.7 (Borel-Cantelli Lemma)**.** *Let $\{E_k\}_{k\geq 1}$ be a collection of measurable sets in $\mathbb{R}^n$ with $\sum_{k=1}^{\infty} m(E_k) < \infty$, then letting $E = \{x \in \mathbb{R}^n | x \in E_k$ for infinitely many $k\}$ is measurable indeed, $m(E) = 0$.*

Proof. If $x \in E$, then for each $i \in \mathbb{N}$ there exists $k \geq i$ such that $x \in E_k$. Translating into set theoretic notations, $E = \cap_{i=1}^{\infty}(\cup_{k\geq i}^{\infty} E_k)$ (define the inside to be F_i). Note that $m(\cup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k) < \infty$ implies that $m(F_i) < \infty$.

Also, $F_i \searrow E$ and so be the preceding prop, $m(F_i) = m(\cup_{k\geq i} E_k) \leq \sum_{k=i}^{\infty} m(E_k) \to 0$. □

3 Lecture 3

Today we will develop basic integration theory in a general setting.

Proposition 3.1. *Let $E \subset \mathbb{R}^n$ be measurable. Then*

1. *$\forall \epsilon > 0$, there exists an open set $O \supset E$ such that $m_(O \setminus E) < \epsilon$.*

2. *There exists F closed such that $F \subset E$, $m(E \setminus F) < \epsilon$*

3. *If $m(E) < \infty$, then there exists a compact $K \subset E$ such that $m(E \setminus K) < \epsilon$*

4. *If $m(E)$ finite, then there exists $F = \cup_{j=1}^{N} Q_j$ with the Q_j closed cubes, such that $m(E \setminus F) + m(F \setminus E) < \epsilon$*

Proof. 1. Straight from definition of measurability.

2. From 1 and complementation.

3. Choose a ball B_k closed for $k \in \mathbb{N}$ such that $m(E \cap B_k) > m(E) - \epsilon/2$. Prossible, since $E \cap B_k \nearrow E$. Then by 2, choose a closed set $F \subset E \cap B_k$ such that $m(E \cap B_k \setminus F) < \epsilon/2$. Then F is compact as desired.

4. Choose a family of closed cubes $\{Q_j\}_{j\geq 1}$ such that $E \subset \cup_{j=1}^{\infty} Q_j$ and $\sum_{j=1}^{\infty} |Q_j| < m(E) + \epsilon/2 < \infty$. Then we can always find N such that $\sum_{j=N+1}^{\infty} |Q_j| < \epsilon/2$.

   Then put $F = \cup_{j=1}^{N} Q_j$ and $m(E \setminus F) + m(F \setminus E) \leq m(\cup_{j=N+1}^{\infty} Q_j) + \sum_{j=1}^{\infty} |Q_j| - m(E) < \epsilon$.

$\square$

Examples of Lebesgue measurable sets $F_\sigma = \cup_{i=1}^{\infty} C_i$ where C_i are closed, $G_\delta = \cap_{i=1}^{\infty} O_i$ where O_i are open. So $F_{\sigma\delta} = \cup_{i=1}^{\infty} F_i$ where the G_i are G_δ sets.

Proposition 3.2. *A subset $E \subset \mathbb{R}^n$ is Lebesgue measurable*

1. *iff E differs from a set in G_δ by a set of measure zero*
2. *iff E differs from a set in F_σ by a set of measure zero.*

Proof. 1. For all $n \geq 1$ choose $O_n \supset E$ open and $m(O_n \setminus E) < 1/n$. Then $\cap O_n \supset E$, and so $0 = m(\cap_{n=1}^{\infty} O_n \setminus E) < 1/k$ for all $k \geq 1$.

2. Follows from 1 by complementation.

$\square$

Definition 3.1 (σ-algebra)**.** *Let X be a set. A collection S of subsets of X is called a σ-algebra provided that*

1. *$\emptyset \in S$*
2. *S is closed under complementation*
3. *S is closed under countable unions.*

Examples: All subspts of X, $\{\emptyset, X\}$, the Lebesgue measureabe subsets of $\mathbb{R}^n$.

Another possible candidate: the Borel algebra.

Definition 3.2 (Borel Algebra)**.** *The Borel σ-algebra is the smallest σ-algebra containing all open subsets of $\mathbb{R}^n$.*

We denote by $\mathscr{L}_{\mathbb{R}^n}$ th eLebesgue measurable sets, $B_{\mathbb{R}^n}$ the Borel σ-algebra, and note that $B_{\mathbb{R}^n}$ is the intersection of all σ-algebras containing the open sets.

Theorem 3.3. *$B_{\mathbb{R}^n} \subsetneq \mathscr{L}_{\mathbb{R}^n} \subsetneq 2^{\mathbb{R}^n}$. That is, there exists a Lebesgue Measurable set that is not Borel, and there exists a non-measurable set.*

The proof is contingent on the axiom of choice.

Lemma 3.4. *Let $E \subset \mathbb{R}^1$ be Lebesgue measurable of positive measure. Then there exists a subset $N \subset E$ which is not LEbesgue measurable.*

Proof. First, we reduce to E bounded, because $E = \cup_{k=1}^{\infty} B_k \cap E$. Then at least one $k \geq 1$ must satisfy $m(E \cap B_k) > 0$. By dilating, we can assume that $E \subset [0,1]$. Define an equivalence relation on E.

Say that $x \sim y$ iff $x-y \in \mathbb{Q}$. For each $x \in E$, denote $E_x = \{y \in E | x-y \in \mathbb{Q}\}$. Choose a maximal set N of inequivalent elements in E. We calim that N is not Lebesgue measurable. We have $E \subset \cup_{k=1}^{N}(N+r_k) \subset [-1,2]$ where $\{r_k\}$ are an enumeration of the rationals in $[-1,1]$. Assume for contradiction that N is measurable. Then so are all the $N + r_k$, and $m(N + r_k) = m(N) \geq 0$. By the disjointedness of the $N + r_k$, if $m(N) > 0$, then $m(\cup(N + r_k)) = \infty$, while if $m(N) = 0$, $m(\cup N + r_k) = 0$, neither of which can occur. $\square$

Now we will prove that $B_{\mathbb{R}^n} \subsetneq \mathscr{L}_{\mathbb{R}^n}$.

Lemma 3.5. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and E a Borel set. Then $\varphi^{-1}(E)$ is also a Borel set.*

Proof. Define $\hat{B}_{\mathbb{R}^n} = \{E \subset \mathbb{R}^n | \varphi^{-1}(E) \text{ is a Borel set}\}$. By the continuity of φ, this contains all the open sets, and for purely set theoretic reasons, $\hat{B}_{\mathbb{R}^n}$ is a σ-algebra, so it contains the Borel algebra. $\square$

Lemma 3.6. *Let $\hat{C}$ and $\tilde{C}$ be two Cantor type sets inside $I = [0,1]$ as in the first lecture, but (using the same notation as in the first lecture) where we also allow $\sum_{k=1}^{\infty} 2^{k-1}\ell_k = 1$. Then, there is a strictly monotonic continuous map $\varphi : I \to I$ such that $\varphi(\hat{C}) \subset \tilde{C}$.*

Proof. φ is obtained as the limit of a bunch of approximations piecewise linear. Now leave φ unchanged on C_k adn continue iteratively, obtaining piecewise linear maps φ_n with $\sup_{x \in [0,1]} |\varphi_n(x) - \varphi_{n+1}(x)| < 1/2^n$, so that the φ_n converge rapidly. Then the limit of the φ_n is a strictly increasing function with $\varphi(\hat{C}) \subset \tilde{C}$.

φ, by continuity and injectivity, is a homeomorphism of I. $\square$

Now we conclude as follows: first arrange that $m(\hat{C}) > 0$ but $m(\tilde{C}) = 0$, this is done by requiring $\sum 2^{k-1}\ell_k < 1$ and $\sum \tilde{\ell}_k 2^{k-1} = 1$.

Now, choose a subset N of $\hat{C}$ which is not Lebesgue measurable. We claim that $\varphi(N)$ is not Borel, but is Lebesgue measurable.

It must be Lebesgue measurable with $m(\varphi(N)) = 0$. So if we assume that $\varphi(N)$ is Borel, then so is $\varphi^{-1}(\varphi(N)) = N$ by injectivity, but this is Borel, and so measurable, contradiction, so the inclusions are all proper.

Definition 3.3 (Abstract Measure Space)**.** *Let X be a set and S a σ-algebra of subsets of X. Then if $m : S \to [0, \infty]$ is a countably additive function with $m(\emptyset) = 0$ we call the triple (X, S, m) a measure space.*

Example: $(\mathbb{R}^n, \mathscr{L}_{\mathbb{R}^n}, m)$, Lebesgue measure.

If we restrict to Borel sets, we get a measure space also.

Now we will develop an integration tehory in this abstract context. Which functions can we integrate? The analogues of step functions, the simple functions, are the correct choices.

At the end of the day, integrable functions are essentially pointwise limits of simple functions.

Definition 3.4 (Measurable). *A function $f : X \to [-\infty, \infty]$ is called measurable iff $f^{-1}((\alpha, \infty])$ is in the σ-algebra of measurable sets for $\alpha \in \mathbb{R}$.*

Remark: Using elementary set theory and σ-algebra properties of S, we can conclude that the above property implies $f^{-1}(\alpha, \beta)$ is measurable and that $f^{-1}([\alpha, \beta])$ is measureable, as is $f^{-1}((\alpha, \beta])$

Lemma 3.7 (Simple Technical Lemma). *Let f_n, $n \geq 1$ be measurable, then so are $\sup f_n$ and $\limsup f_n$.*

Proof. For sup, let $g = \sup f_n$. Then $g^{-1}((\alpha, \infty]) = \cup_{n=1}^{\infty} f_n^{-1}((\alpha, \infty])$, and so done. Note that $\inf f_n = -\sup(-f_n)$. Then for $\limsup f_n$, note that it is $\inf \sup f_n$. $\square$

Corollary 3.8. *Letting $f_+ = \max\{f, 0\}$ and $f_- = \min\{f, 0\}$, then f is measurable iff $f_\pm$ is measurable.*

Definition 3.5 (Simple). *A function $f : X \to [0, \infty)$ is called simple provided that it attains only finitely many values.*

In particular, we can write $f(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$ if f is also measurable and A_i is measurable, and $\chi_{A_i}(x)$ is the characteristic function of A_i.

Lemma 3.9. *Let $f : X \to [0, \infty]$ be measurable. Then there exist simple measurable functions s_n on X such that $0 \leq s_1 \leq s_2 \leq \ldots \leq f$ and $f(x) = \lim s_n(x)$.*

Conversely, if f is defined by such a limit, by the preceding lemma, f is measurable.

Proof. Choose $n \geq 1$ and for $1 \leq i \leq n2^n$ define $E_{n,i} = f^{-1}([\frac{i-1}{2^n}, \frac{i}{2^n}])$ and $F_n = f^{-1}([n, \infty])$. Then define $s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} + n\chi_{F_n}$.

To check the monotonicity of $\{s_k\}$, note that passing from n to $n+1$ we splie each $E_{n,i}$ into 2 halves and replace $\frac{i-1}{2^n}\chi_{E_{n,i}}$ by $\frac{i-1}{2^n}\chi_{E_{n+1,2i-1}} + \frac{2i-1}{2^{n+1}}\chi_{E_{n+1,2i}} \geq \frac{i-1}{2^n}\chi_{E_{n,i}}$.

Further, it is easy to check that $s_n(x) \to f(x)$ pointwise (if $f(x) < \infty$, then $s_n(x) \geq f(x) - 2^{-n}$ and if $f(x) = \infty$, then $s_n(x) \geq n$ for n large enough in both cases) $\square$

4 Lecture 4

Integration theory on general measure spaces (X, S, μ). Last time we defined measurable functions.

Definition 4.1 (Integral). *Let f be a simple function. Then*

$$\int_E f d\mu = \sum_{i=1}^{N} a_i \mu(E \cap A_i)$$

.

From here, we can define the integral of a nonnegative measurable function $f : X \to [0, \infty]$ as follows:

$\int_E f d\mu = \sup_{s \leq f} \int_E s d\mu$. To pass from nonnegative measurable functions to general function $f : X \to [-\infty, \infty]$, set $\int_E f d\mu = \int_E f_+ d\mu - \int_E f_- d\mu$ provided that one of the two integrals on the right is finite.

Then, if f is complex valued, we define $\int_E f d\mu$ in a real and imaginary part.

The core of integration theory consists of three theorems:

1. Monotone Convergence Tehorem
2. Fatou's Theorem
3. Lebesgue Dominated Convergence Theorem

Theorem 4.1 (Monotone Convergence)**.** *Suppose that* $0 \leq f_1(x) \leq f_2(x) \leq \ldots \leq \infty$ *such that* $\lim_{n\to\infty} f_n(x) = f(x)$ *for all* $x \in X$*, then*

$$\lim_{n\to\infty} \int_X f_n(x) d\mu = \int_X f(x) d\mu$$

Remark: Nonnegativity is crucial!

Proof. By monotonicity, the limit $\lim_{n\to\infty} \int_X f_n(x) d\mu = A$ exists in $[0, \infty]$. Further, f is measurable, and by the monotonicity of $\int_X f d\mu$ with respect to f, $A \leq \int_X f d\mu$. We need to get the other inequality.

Now choose a simple measurable function s, $0 \leq s \leq f$ and choose a number $\theta \in (0, 1)$ which we eventually let go to 1. Then introduce the sets $E_n = \{x \in X | f_n(X) \geq \theta s(x)\}$. Then $E_1 \subset E_2 \subset \ldots$ and $\cup_{n=1}^{\infty} E_n = X$.

So now $\int_X f_n d\mu \geq \int_{E_1} d_n d\mu \geq \int_{E_n} \theta s(x) d\mu$ and now let $n \to \infty$. We get $\int_{E_n} \theta s(x) d\mu \to \int_X \theta s(x) d\mu$. Now, since $\theta < 1$ was arbitrary, $A \geq \int_X s(x) d\mu$. $\square$

Theorem 4.2 (Fatou)**.** *Let* $f_n : X \to [0, \infty]$ *measurable, then* $\int_X \liminf f_n d\mu \leq \liminf \int_X f_n(x) d\mu$*.*

Proof. This follows from Monotone Convergence. $\liminf_{n\to\infty} f_n = \lim_{k\to\infty} \inf_{1\geq k} f_i = \lim_{k\to\infty} g_k$.

Then $0 \leq g_1 \leq g_2 \leq \ldots$ and hence by monotone convergence, we have $\int_X \liminf f_n d\mu = \lim_{k\to\infty} \int_X g_k d\mu$. As $g_k \leq f_k$, we have $\leq \liminf \int_X f_n d\mu$. $\square$

Definition 4.2 (L^1)**.** *Let* (X, S, μ) *be a measure space. Then* $L^1(d\mu)$ *denotes the set of all measurable functions such that* $\int_X |f|(x) d\mu < \infty$

For this to make sense, need the following:

Lemma 4.3. *Let* $f : X \to [-\infty, \infty]$ *be measurable and* $\varphi : [-\infty, \infty] \to [-\infty, \infty]$ *continuous. Then* $\varphi \circ f$ *is measurable.*

Proof. Need to check that $(\varphi \circ f)^{-1}((\alpha, \infty])$ is measurable, that is, $f^{-1}(\varphi^{-1}((\alpha, \infty])$ is. $(\alpha, \infty] = \cup(\alpha, n)$ and $\varphi(\alpha, n)$ is open, by continuity. $\square$

Theorem 4.4 (Dominated Convergence)**.** *Assume that $f_n : X \to [-\infty, \infty]$ is measurable and $|f_n|(x) \le g(x)$ for all $x \in X$ where $g \in L^1(d\mu)$. Then if $f_n \to f$ pointwise, then $\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0$. One says that $f_n \to f$ in L^1.*

In particular, $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.

Proof. Application of Fatou, by triangle inequality. $|f_n - f| \le 2g$. So $\int_X 2gd\mu \le \liminf \int (2g - |f_n - f|)d\mu = \int_X 2gd\mu + \liminf \int_X -|f_n - f|d\mu = \int_X 2gd\mu - \limsup \int_X |f_n - f|d\mu$.

Hence, $0 \le \limsup \int_X |f_n - f|d\mu \le 0$, and so $\lim_{n\to\infty} \int_X |f_n - f|d\mu = 0$.

$\left|\int_X f_n d\mu \int f d\mu\right| \le \int_X |f_n - f|d\mu \to 0$. □

Simple Application

Theorem 4.5. *Let $f : I \to \mathbb{R}$ be Riemann Integrable. Then f is measurable and $\int_{Riemann} f(x)dx = \int_I f(X)d\mu$.*

Proof. By definition of Riemann integrability, there exist sequences of functions φ_i, ψ_i step functions the φ_i increasing from $-M$ to f and the ψ_i decreasing from f to M. To apply the LDCT, put $g = M\chi_I$. Then $g \in L^1(d\mu)$ and $|\varphi_i| \le g$ and $|\psi_i| \le g$, and as the $\varphi_n \to f$ pointwise and $\psi_n \to f$ pointwise, we have $\int f(x)dx = \lim_{n\to\infty} \int \varphi_n(x)dx = \lim_{n\to\infty} \int \varphi_n(x)d\mu = \int f(x)d\mu$ □

Let (X, S, μ) be a measure space and f a function. Then $\int_X f d\mu \in \mathbb{R}$, and it is well-defined in $f \in L^1(d\mu)$ and it is linear, so $\int_X d\mu$ is a vector space homomorphism from $L^1(d\mu) \to \mathbb{R}$.

Can one go the other way around? Given a homomorphism, find μ?

Let $(X, S, \mu) = (\mathbb{R}^n, B_{\mathbb{R}^n}, \mu)$. Then $L^1(\mathbb{R}^n)$ contains the continuous, compactly supported functions.

Theorem 4.6 (Riesz Representation Theorem)**.** *Let $\Lambda : C_c(\mathbb{R}^n) \to \mathbb{C}$ be a homormophism from compactly supported continuous functions to $\mathbb{C}$. Assume that if $f \ge 0$ then $\Lambda f \ge 0$. Then there exists a unique Borel measure μ (a measure defined on a σ-algebra containing the Borel sets) such that*

1. *$\Lambda f = \int_{\mathbb{R}^n} f d\mu$*
2. *$\mu(K) < \infty$ for all $K \subset \mathbb{R}^n$ compact*
3. *μ is almost regular in the following sense: For $E \in B_{\mathbb{R}^n}$, $\mu(E) = \inf\{\mu(V) | E \subset V, V \text{ open }\} = \sup\{\mu(K) | K \subset E \text{compact}\}$ provided that $\mu(E) < \infty$ or E open.*

We will need two lemmas.

Lemma 4.7 (Urysohn's Lemma)**.** *Let $A \subset \mathbb{R}^n$ compact and $B \subset \mathbb{R}^n$ closed with $A \cap B = \emptyset$. Then there exists a function $f \in C_c(\mathbb{R}^n)$ with $f \equiv 1$ on A and $f \equiv 0$ on B.*

Proof. $d(A, B) > 0$. Find a cont. function $\varphi : [0, \infty] \to [0, 1]$ such that $\varphi \equiv 1$ if $0 \le x \le d(A, B)/2$ and $\varphi(x) = 0$ for $x \ge \frac{2}{3}d(A, B)$. Then put $f(x) = \varphi(d(x, A))$ □

Lemma 4.8 (Partition of Unity)**.** *Let $K \subset \mathbb{R}^n$ a compact set, and $\cup_{i=1}^N V_i \supset K$ a finite open covering. Then there exist cont. functions h_i such that* $\operatorname{supp} h_i \subset V_i$ *and* $\sum_{i=1}^N h_i \equiv 1$.

Proof. Choose $U_i \subset V_i$ such that $\bar{U_i} \subset V_i$ and $\cup U_i \supset K$. Then apply Urysohn to find $\tilde{h}_i$ such that $\tilde{h}_i \equiv 1$ on $\bar{U_i}$ and 0 outsice of V_i.

Now we define h_i such that $h_1 = \tilde{h}_1$, $h_2 = (1 - \tilde{h}_1)\tilde{h}_2$, etcetera. Then the sum is 1 on K. □

5 Lecture 5

Theorem 5.1 (Riesz Representation Theorem)**.** *Let $\Lambda : C_c(\mathbb{R}^n) \to \mathbb{C}$ be a linear functional such that $\Lambda(f) \geq 0$ if $f \geq 0$ then there exists a unique Borel measure m such that*

1. *$\Lambda f = \int_{\mathbb{R}^n} f dm$*
2. *$m(K) < \infty$ if U is compact*
3. *Partial Regularity: $m(E) = \inf\{m(V) | E \subset V$ open $\}$ and $m(E) = \sup\{m(K) | E \supset K$ compact$\}$ if $m(E) < \infty$ or E is open.*

Proof. First we will prove uniqueness. Assume that m_1 and m_2 both satisfy the conditions. If suffices to show that $m_1(K) = m_2(K)$ for all K compact by partial regularity. We know that these numbers are finite, and we also know that for all $\epsilon > 0$, there exists $V \supset K$ open such that $m_2(V) < m_2(K) + \epsilon$. By Urysohn, there exists a function f which is compactly supported and continuous such that $V \supset \operatorname{supp} f$ and $f \equiv 1$ on K. Then $m_2(K) + \epsilon \geq m_2(V) = \int_{\mathbb{R}^n} \chi_V dm_2 \geq \int f dm_2 = \Lambda f = \int f dm_1 \geq \int_{\mathbb{R}^n} \chi_K dm_1 = m_1(K)$, so for all $\epsilon > 0$, $m_2(K) + \epsilon \geq m_1(K)$, and so $m_2(K) \geq m_1(K)$ and symmetry gives us $m_1(K) = m_2(K)$.

Now we will show existence. Given V open, $m(V) = \int \chi_V dm$. Define $m(V) = \sup_{f \in C_c(\mathbb{R}^n)} \Lambda f$ over $0 \leq f \leq 1$ and $\operatorname{supp} f \subset V$.

Define something like the outer measure, $E \subset \mathbb{R}^n$ any subset, let $m(E) = \inf_{V \supset E} m(V)$ for V open.

First goal is to show that m is well-behaved on compact sets.

We must show sub-additivity. $\{E_i\}_{i=1}^\infty \subset \mathbb{R}^n$, we want to show that $m(\cup E_i) \leq \sum m(E_i)$.

We may assume that $m(E_i) < \infty$, and so given $\epsilon > 0$, there exist open sets $V_i \supset E_i$ such that $m(V_i) < m(E_i) + \epsilon/2^i$, and $\cup V_i \supset \cup E_i$.

So $m(\cup V_i) < m(\cup E_i) + \epsilon$, and so we apply the construction of m on oepn sets and pick $f \in C_c(\mathbb{R}^n)$ with $\operatorname{supp} f \subset \cup V_i$, which gives us a finite collection $f_1, \ldots, f_N$ such that $\operatorname{supp} f \subset \cup_{i=1}^N V_i$. Now we apply a partition of unity to $K = \operatorname{supp} f$ and get $h_i \in C_c(\mathbb{R}^n)$ with $\operatorname{supp} h_i \subset V_i$, $0 \leq h_i \leq 1$ and $\sum h_i|_K = 1$.

So then $\Lambda f = \sum_{i=1}^N \Lambda(h_i f) \leq \sum_{i=1}^N m(V_i) \leq \sum_{i=1}^\infty m(V_i)$, so we take the supremum over all f and then $m(\cup V_i) \leq \sum m(V_i) \leq \sum m(E_i) + \epsilon$. Letting $\epsilon \to 0$, we get $m(\cup E_i) \leq \sum m(E_i)$.

So m behaves like an outer measure so far. We need to find a σ-algebra of sets containing the Borel sets, on which it is a measure.

$\tilde{S} = \{E \subset \mathbb{R}^n | m(E) < \infty, m(E) = \sup_{K \subset E}\{m(K)\}\}$. We will show that this is an algebra and contains all open sets of finite measure and all compact sets.

Then we take $S = \{E \subset \mathbb{R}^n | E \cap K \in \tilde{S}$ for all K compact$\}$.

Main Assertion: S is a σ-algebra containing $B_{\mathbb{R}^n}$ and m is a measure on S, and $\Lambda f = \int f dm$ for $f \in C_c(\mathbb{R}^n)$.

If K is compact, then $K \in \tilde{S}$ and $m(K) = \inf\{\Lambda f | f = 1 \text{ on } K\}$. We prove this by choosing f to eb 1 on K and $0 \leq f \leq 1$, $f \in C_c(\mathbb{R}^n)$. We can do this by Urysohn. Then $\Lambda f \geq m(K)$. Fix $0 < \theta < 1$ and $V_\theta = \{x | f(x) > \theta\} \supset K$. If $g \in C_c(\mathbb{R}^n)$, $\operatorname{supp} g \subset V_\theta$, $0 \leq g \leq 1$ then $\theta g < f$ and $m(K) \leq m(V_\theta) = \sup\{\Lambda g | \operatorname{supp} g \subset V_\theta, 0 \leq g \leq 1\} = \sup\{\theta^{-1}\Lambda\theta g\} \leq \theta^{-1}\Lambda f$. Now we let $\theta \to 1$ and get $m(K) \leq \Lambda f$ and so $m(K) < \infty$.

By outer regularity, we have $\forall \epsilon > 0$, there exists an open set V such that $m(K) + \epsilon > m(V)$. BY Urysohn, $f \in C_c(\mathbb{R}^n)$, $f \equiv 1$ on K, $\operatorname{supp} f \subset V$.

$m(K) \leq \Lambda f < m(K) + \epsilon$, and so $m(K) = \int_{f|_K \equiv 1} \Lambda f$.

Now we must show finite additivity on compact sets. It suffices to let $K_1, K_2 \subset \mathbb{R}^n$ compact and disjoint. Then $m(K_1) + m(K_2) = m(K_1 \cup K_2)$. This is because there is an $f \in C_c(\mathbb{R}^n)$ that is 1 on $K_1 \cup K_2$ and $\Lambda f \leq m(K_1 \cup K_2) + \epsilon$, by Urysohn, there exist $h_1, h_2 \in C_c(\mathbb{R}^n)$ such that $h_i \equiv 1$ on K_i and $h_i \equiv 0$ on K_j, $i \neq j$. Then $\Lambda f = \Lambda(fh_1) + \Lambda(fh_2)$ Then for all $\epsilon > 0$, $m(K_1 \cup K_2) + \epsilon \geq m(K_1) + m(K_2)$, but $m(K_1 \cup K_2) \leq m(K_1) + m(K_2)$, and so they must be equal.

From here, we need to get countable additivity on $\tilde{S}$. Assume $E = \cup E_i$ are disjoint and $E_i \in \tilde{S}$. Then we claim that $m(E) = \sum m(E_i)$ and if $m(E) < \infty$ then $E \in \tilde{S}$.

We use inner regularity of m on $\tilde{S}$. For all $\epsilon > 0$, choose $H_i \subset E_i$ and $m(E_i) < m(H_i) + \epsilon/2^i$ for all i. Then by te last part, each finite sum is equal. Then $m(E) \geq m(\cup_{i=1}^N H_i) = \sum_{i=1}^N m(H_i) > \sum m(E_i) - \epsilon$.

Now let $N \to \infty$ and we get $m(E) \geq \sum m(E_i) - \epsilon$ and letting $\epsilon \to 0$ get $m(E) \geq \sum m(E_i)$. Subadditivity gives equality.

Now we need to show that m is well behaved on open sets. If $E \subset \mathbb{R}^n$ is open, then $m(E) = \sup\{m(K) | K \subset E \text{compact}\}$. In partiacular, E is open and $m(E) < \infty$ imply $E \in \tilde{S}$.

Choose a number $\theta < m(E)$. Then there exists $f \in C_c(\mathbb{R}^n)$ with $\operatorname{supp} f \subset E$ and $\theta < \Lambda f$. Then let $K = \operatorname{supp} f$. We want to show that $m(K) > \theta$. $m(K) = \inf\{m(W)\}$ for $W \supset K$, then $m(W) \geq \Lambda f$.

Refined version of regularity of m: if $E \in \tilde{S}$ then there exists $K \subset E$ and $V \supset E$ such that $m(V \setminus K) < \epsilon$.

By outer regularity, there exists $V \supset K$ such that $m(K) + \epsilon > m(V)$. By inner regularity, there exists $K \subset E$ compact such that $m(E) < m(K) + \epsilon$, and so $m(V) < \infty$ implies that $V \in \tilde{S}$ and $K \in \tilde{K}$, so $V \setminus K \in \tilde{S}$. By the additivity of m on $\tilde{S}$, $m(K) + m(V \setminus K) = m(V) < m(E) + \epsilon$ and so $m(V \setminus K) < 2\epsilon$.

We must now show that $\tilde{S}$ is an algebra.

Let $A, B \in \tilde{S}$. By the last step, there exist $K_1 \subset A \subset V_1$ and $K_2 \subset B \subset V_2$

such that $m(V_1 \setminus K_1) < \epsilon$ and $m(V_2 \setminus K_2) < \epsilon$. Then $A \setminus B \subset V_1 \setminus K_2 \subset V_1 \setminus K_1 \cup K_1 \setminus V_2 \cup V_2 \setminus K_2$, and so $m(A \setminus B) \leq 2\epsilon + m(K_1 \setminus V_2)$. Now, $K_1 \setminus V_2 \subset A \setminus B$ is a compact subset, and since $\epsilon \to 0$ arbitrary, we get inner regularity, and so $m(A \setminus B) < \infty$ implies that $A \setminus B \in \tilde{S}$.

$A \cup B = A \setminus B \cup B \in \tilde{S}$, and $A \cap B = A \setminus (A \setminus B)$.

Now we show that S is a σ-algebra containing the Borel sets.

If $A \in S$ then $A \cap K \in \tilde{S}$ for all K compact, $A^c \cap K = K \setminus (K \setminus A^c) = K \setminus (K \cap A) \in \tilde{S}$. Hence $A^c \in S$.

Next let $A_i \in S$, we must shw that $\cup A_i \in S$. $A \cap K = A_1 \cap K \cup (A_2 \cap K) \setminus (A_1 \cap K) \cup \ldots$ and by inductively applying the algebraic properties of $\tilde{S}$, we see that each of the sets in the union are in $\tilde{S}$, and are disjoint. Also, $m(A \cap K) \leq m(K) < \infty$, and so countable additivity of m on $\tilde{S}$ gives us that $A \cap K \in \tilde{S}$.

Claim: $\tilde{S} = \{E \in S | m(E) < \infty\}$.

Clearly $\tilde{S} \subset S$, now assume that $E \in S$ and has finite measure. Choose $V \supset E$ open with $m(E)+\epsilon > m(V)$. Then choose $K \subset V$ such that $m(V \setminus K) < \epsilon$ by the inner regularity of m. Then $E \cap K \in \tilde{S}$ and so $K_1 \subset E \cap K$ such that $m(E \cap K) \leq m(K_1) + \epsilon$. Finally, $E \subset E \cap K \cup (V \setminus K)$. Ad so $m(E) \leq m(E \cap K) + m(V \setminus K) \leq m(K_1) + 2\epsilon$. And so $E \in \tilde{S}$.

Now we must show that m is a measure on S.

If $E = \cup E_i$ disjoint, then if $m(E_i) = \infty$ for some i, then $m(E) = \infty = \sum m(E_i)$, hence assume $m(E_i) < \infty$ for all i, then $E_i \in \tilde{S}$ and so m is countably additive on S.

All that remains is to show that m represents Λ. □

6 Lecture 6

We will finish the proof of the Riesz Representation Theorem.

We have $\Lambda : C_c(\mathbb{R}^n) \to \mathbb{C}$ with $\Lambda f \geq 0$ whenever $f \geq 0$ when f is a function into $\mathbb{R}$.

If E is open, then $m(E) = \sup_{\operatorname{supp} f \subset E}\{\Lambda f\}$ and fr E in general, we have $m(E) = \inf m(V)$ where $V \supset E$ open.

$\tilde{S} = \{E \subset \mathbb{R}^n | m(E) < \infty, m(E) = \sup m(K)$ where $K \subset E$ compact$\}$. So we expand to $S = \{E \subset \mathbb{R}^n | E \cap K \in \tilde{S}$ for all K compact$\}$. We showed that $m|_S$ is a measure and that S is a σ-algebra.

Lemma 6.1. *$\Lambda f = \int_{\mathbb{R}^n} f dm$ for all $f \in C_c(\mathbb{R}^n)$*

Proof. Suffices to show that $\Lambda f \leq \int_{\mathbb{R}^n} f dm$ for all f, and then also $\Lambda(-f) \leq \int_{\mathbb{R}^n} (-f) dm$ gives us $\Lambda f \geq \int_{\mathbb{R}^n} f dm$.

Approximate Λf be a "Riemann Sum", choose $[a, b] \supset \operatorname{range}(f)$ and choose φ_i by $\varphi_0 < a < \varphi_1 < \ldots < \varphi_n = b$ with $\varphi_i - \varphi_{i-1} < \epsilon$, $E_i = f^{-1}((\varphi_{i-1}, \varphi_i]) \cap \operatorname{supp} f$ is a Borel set. Then $V_i \supset E_i$, $m(V_i) < m(E_i) + \epsilon/n$.

$f(x) < \varphi_i + \epsilon$, $\forall x \in V_i$. By the partition of unity lemma, we can find $h_i \in C_c(\mathbb{R}^n)$ supported inside V_i which sum to 1 on the support of f. And so $m(\operatorname{supp} f) \leq \Lambda(\sum h_i)$.

$\Lambda f = \sum \Lambda(f h_i) \leq \sum \Lambda(h_i(\varphi+\epsilon)) = \sum(\varphi_i+\epsilon)h_i = \sum(\varphi)_i + \epsilon + |a|)\Lambda h_i - |a| \sum \Lambda(h_i)$.

So we have $\Lambda(h_i) \leq m(V_i) < m(E_i) + \epsilon/n$, so we have $\leq \sum(|a| + \varphi_i + \epsilon)(m(E_i) + \epsilon/n) - |a| m(suppf) = \sum(\varphi_i - \epsilon)m(E_i) + 2\epsilon m(\text{supp}\, f) + \epsilon/n \sum(\epsilon + |a| + \varphi_i)$. The first term is less than $\int_{\mathbb{R}^n} f dm$, and so when we let $\epsilon \to 0$, we are done. $\square$

L^p spaces. Let (X, S, m) be a measure space.

Let $f : X \to \mathbb{C}$ mearuable and $1 \leq p < \infty$. Then $|f|^p \in L^1(dm)$ tells us that $f \in L^p(dm)$, and put $\|f\|_{L^p} = \left(\int_X |f|^p dm\right)^{1/p}$. If $f \in L^p$, and g differs from f on a set of measure 0, then $g \in L^p$, and $\int_X |f-g|^p dm = 0$, so what we really want in L^p are equivalence classes of functions such that $f \sim g$ iff $f - g = 0$ almost everywhere.

Definition 6.1 (L^p)**.** *$L^p(dm)$ is teh set of equivalence classes of functions.*

Definition 6.2 (L^∞)**.** *$L^\infty(dm)$ is the set of all equivalence classes of measurable functions $f : X \to \mathbb{C}$ such that $\inf_{E \subset X, m(E)=0} \sup_{X \setminus E} |f| < \infty$.*

In particular, there exists a set $\tilde{E}$ of measure 0 such that $\sup_{X \setminus \tilde{E}} |f| < \infty$.

Theorem 6.2. *$L^p(dm)$ is a vector space for $1 \leq p \leq \infty$.*

Proof. Clear if $p = 1, \infty$.

For the other cases, it follows immediately from Minkowski Inequality, which says that if $f, g \in L^p(dm)$ then $\|f + g\| \leq \|f\| + \|g\|$ in L^p norm. This follows from Hölder's Inequality, which says that if $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then $\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^p}$.

We assume that p, q not $1, \infty$. Then we use the elementary inequality $a, b \in \mathbb{C}$ then $|ab| \leq |a|^p/p + |b|^q/q$. We assume that $a, b \in \mathbb{R}_{>0}$, then we get $\log(ab) = \frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q) \leq \log(a^p/p + b^q/q)$.

Given $f \in L^p(dm)$, $g \in L^q(dm)$, neither equal to zero, then $|fg| \leq |f|^p/p + |g|^q/q$. Then dividing by normas, we gat $|f/\|f\|, g/\|g\|| \leq |f|^p/p\|f\|^p + |g|^q/q\|g\|^q$. This gives us that $\|fg\|_{L^1}/\|f\|_{L^p}\|g\|_{L^q} \leq \int |f|^p dm/p\|f\|^p_{L^p} + \int |g|^q/q\|g\|^q_{L^q} = 1$.

So now we prove Minkowski: $1 < p < \infty$, $\int_X |f+g|^p dm \leq \int(|f| + |g|)^p dm = \int |f|(|f| + |g|)^{p-1} dm + \int |g|(|f| + |g|)^{p-1} dm$. Apply Hölder to each for $p, q = \frac{p}{p-1}$. Then $\int |f|(|f| + |g|)^{p-1} dm \leq \|f\|_{L^p}\left(\int_X [(|f| + |g|)^{p-1}]^{p/p-1} dm\right)^{p-1/p} = \|f\|_{L^p}\left(\int_X (|f| + |g|)^p dm\right)^{p-1/p} = \|f\|_{L^p} + \||f| + |g|\|^{p-1}_{L^p}$, and similarly for g. So then $\||f| + |g|\|^p_{L^p} \leq (\|f\|_{L^p} + \|g\|_{L^p})\||f| + |g|\|^{p-1}_{L^p}$. $\square$

Corollary 6.3. *$L^p(dm)$ is a metric space with distance function $d(f, g) = \|f - g\|_{L^p}$.*

Theorem 6.4. *$1 \leq p \leq \infty$, $L^p(dm)$ is a complete metric space.*

Proof. First we will do this for $p = \infty$. Let $\{f_n\}$ be a Cauchy sequence in L^∞. Then $\|f_n - f_m\| \geq |(f_n - f_m)|$ for x outside a set $B_{n,m}$...GAHSHSHAHASHAS

$Y = \cup_{n,m} B_{n,m} \cup \cup_n A_n$. Then $m(Y) = 0$. For $x \in X \setminus Y$, $f_n(x) \to f$, f measurable. $\|f_n \to f\|_{L^\infty} \to 0$.

Remark: f is only defined on $X \setminus Y$, but one can extend it arbitrarily on Y.

Now choose a finite p. Let $\{f_n\}$ be a Cauchy sequence in L^p. Choose a subseqence $\{f_{n_k}\}_{k\geq 1}$ such that $\|f_{n_k} - f_{n_{k-1}}\| \leq \frac{1}{2^k}$. Take $g_k = \sum_{i=1}^k |f_{n_{i+1}} - f_{n_i}|$, $g = \sum_{i=1}^\infty |f_{n_{i+1}} - f_{n_i}|$. By Minkowski, we have that $\|g_k\|_{L^p} < 1$, and so Fatou's Lemma tells us that $\|g\|_{L^p} \leq 1$.

In particular, $g(x) < \infty$ ae. Then $f_1(x) + \sum_{i=1}^\infty (f_{n_{i+1}} - f_{n_i})$ converges ae, and so we can define $f(x)$ to be the limit ae.

By Fatou's Lemma again, $\int_X |f - f_n|^p dm \leq \liminf \int_X |f_{n_i} - f|^p dm$. If we take $(f_1(x) + \sum_{i=1}^k (f_{n_{i+1}} - f_{n_i})$ and $n_1 = 1$, we have $f_{n_{k+1}}$.

As it is Cauchy, if $n \to \infty$, $\int_X |f - f_n|^p dm \to 0$. $\square$

$p = 2$, $L^2(dm)$ is special. $f, g \in L^2(dm)$, then we have $\int f\bar{g} dm = \langle f, g\rangle$ has absolute value $\|f\|_{L^2}\|g\|_{L^2}$, by Cauchy-Schwartz. Completeness means that L^2 is a Hilbert Space, not just a Banach Space.

Definition 6.3 (Hilbert Space)**.** *A vector space H over $\mathbb{C}$ or $\mathbb{R}$ equipped with a Hermitian inner product $\langle -, -\rangle : H \times H \to \mathbb{C}$ such that $\langle af, bg\rangle = a\bar{b}\langle f, g\rangle$. and $\langle f, f\rangle \geq 0$ with $\langle f, f\rangle = 0$ iff $f = 0$ is called a Hilbert Space provided it is complete with respect to the metric $d(f, g) = \langle f - g, f - g\rangle^{1/2}$.*

Remark: Fact that this is a metric follows from Cauchy-Schwartz inequality.

We still need Fubini's Theorem:

Theorem 6.5 (Fubini)**.** *Write $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$. Assume that $f \in L^1(\mathbb{R}^n)$. In particular, f is measurable. Then for almost every $y \in \mathbb{R}^{n_2}$ the function $f^y(x) = f(x, y)$ is integrable, and the function $y \mapsto \int_{\mathbb{R}^{n_1}} f^y(x)dx$ is measurable (defined outside a measure zero set). In addition,*

$$\int_{\mathbb{R}^{n_1}} (\int_{\mathbb{R}^{n_1}} f^y(x)dx)dy = \int_{\mathbb{R}^n} f(x, y)dxdy$$

7 Lecture 7

Theorem 7.1 (Fubini)**.** *Write $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$. Assume that $f \in L^1(\mathbb{R}^n)$. In particular, f is measurable. Then for almost every $y \in \mathbb{R}^{n_2}$ the function $f^y(x) = f(x, y)$ is integrable, and the function $y \mapsto \int_{\mathbb{R}^{n_1}} f^y(x)dx$ is measurable (defined outside a measure zero set). In addition,*

$$\int_{\mathbb{R}^{n_1}} (\int_{\mathbb{R}^{n_1}} f^y(x)dx)dy = \int_{\mathbb{R}^n} f(x, y)dxdy$$

Proof. Start with simple function. Let's call the set of all functions for which Fubini holds $\mathscr{F}$.

1. $\mathscr{F}$ is closed under finite linear combinations.

2. Closed under monotone limits.

   By passing to $\{-f_k\}_{\geq 0}$, if suffices to assume that $f_k \uparrow f$. Then $f_k \mapsto f_k - f_1$ along with the first property tell us that we can assume $f_k \geq 0$. Hence, we assume $f_k \uparrow f$ and $f_k \geq 0$. By Monotone Convergence Theorem, $\int_{\mathbb{R}^n} f_k(x,y)dxdy \to \int_{\mathbb{R}^n} f(x,y)dxdy$.

   For all $k \geq 1$, there exists A_k with measure 0 such that $f_k^y(x)$ is integrable if $y \notin A_k$. $A = \cup A_k$, and $m(A) = 0$. If $y \notin A$, then we can define $g_k(y) = \int_{\mathbb{R}^n} f_k^y(x)dx$ for each $k \geq 1$, and by assumumption $\int_{\mathbb{R}^m} g_k(y)dy = \int_{\mathbb{R}^n} f_k(x,y)dxdy \to \int_{\mathbb{R}^n} f(x,y)dxdy$...etc

3. Characteristic function of a G_δ set is in $\mathscr{F}$.

   First check this for open cubes E. Then $E = Q_1 \times Q_2 \subset \mathbb{R}^n \times \mathbb{R}^m$. $g(y) = \int_{\mathbb{R}^n} \chi_E(x,y)dx$ if $|Q_1|$ if $y \in Q_2$ and 0 else. And $g(y) = \chi_{Q_2}|Q_1|$. Then $\int_{\mathbb{R}^m} g(y)dy = |Q_1||Q_2| = \int_{\mathbb{R}^n} \chi_E dxdy$... CONTINUE ON

4. if $m(E) = 0$, then $\chi_E \in \mathscr{F}$.

   Choose a G_δ set $G \supset E$ with $m(G) = 0$. By part 3, $\chi_G \in \mathscr{F}$. $\int_{\mathbb{R}^{n_2}} dy \int_{\mathbb{R}^{n_1}} \chi_G(x,y)dx = \int_{\mathbb{R}^n} \chi_G(x,y)dxdy = 0$ Now $E^y = \{x|(x,y) \in E\}$ is contained in G^y, and $m(G^y) = 0$ for ae $y \in \mathbb{R}^{n_2}$, so $m(E^y) = 0$ for ae $y \in \mathbb{R}^{n_2}$.

   $\int_{\mathbb{R}^{n_2}} dy \int_{\mathbb{R}^{n_1}} \chi_E(x,y)dxdy = 0 = \int_{\mathbb{R}^n} \chi_E(x,y)dxdy$.

5. E is measurable, of finite measure implies that $\chi_E \in \mathscr{F}$

6. $f \in L^1(\mathbb{R}^n)$ then $f \in \mathscr{F}$.

□

Let $1 \leq p \leq \infty$, look at $L^p(\mathbb{R}^n)$.

Theorem 7.2. *$C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.*

Proof. Split $f = f_+ - f_-$. Approximate f_+, f_- by simple meaureable functions s_1, s_2 with $0 \leq s_1 \leq f_+$ and $0 \leq s_2 \leq f_-$ such that $\int_{\mathbb{R}^n} |f_+ - s_1|^p dx < \epsilon$ and $\int_{\mathbb{R}^n} |f_- - \epsilon|^p dx < \epsilon$.

Etc □

Theorem 7.3. *$C_c^\infty(\mathbb{R}^n)$ is dence in $L^p(\mathbb{R}^n)$.*

Result: $D^\alpha(\varphi * f) = D^\alpha \varphi * f \in C^\infty(\mathbb{R}^n)$ where D^α =product of partials.

Lemma 7.4. *Let $f \in C^0(\mathbb{R}^n)$ and $\varphi_h = h^{-n}\varphi(x/h)$, φ as before. Then $\lim_{h\to 0} \varphi_h * f = f$ uniformly on compact subsets of $\mathbb{R}^n$*

Proof. $\varphi_h * f = \int_{\mathbb{R}^n} \varphi(z)f(x-hz)dz$, the function $z \to f(x-hz)$ converges uniformly toward $f(x)$ for x varying over compact subsets of $\mathbb{R}^n$. So $\lim_{h\to 0} \int \varphi(z)f(x-hz)dz = \int_{\mathbb{R}^n} \varphi(z)f(x)dz = f(x)$

Moreover, convergence uniform for x confined to compact subsets of $\mathbb{R}^n$. □

We can now prove that $C_c^\infty(\mathbb{R}^n)$ is dense in L^p.

Given $f \in L^p$, and $\epsilon > 0$, find $\tilde{g} \in C_c^\infty$ such that $\|f - \tilde{f}\|_{L^p} < \epsilon$.

Goal: Show that $\varphi_h * f \to h$ in L^p.

$\|\varphi_h * f - f\|_{L^p} \leq \|\varphi_h * (f - \tilde{f})\|_{L^p} + \|\varphi_h * \tilde{h} - \tilde{f}\|_{L^p} + \|f - \tilde{f}\|_{L^p}$.

First, choose h small enough such that $\|\varphi_h * \tilde{f} - \tilde{f}\|_{L^p} < \epsilon/2$.

For $\varphi_h * (f - \tilde{f})$, we'll show that it has smaller L^p norm than $f - \tilde{f}$ for all $h > 0$.

$\varphi_h * g = \int_{\mathbb{R}^n} \varphi(z) g(x - hz) dz = \int_{\mathbb{R}^n} \varphi^{1/p}(z) \varphi^{1/q}(z) g(x - hz) dz$ where $1/p + 1/q = 1$.

By Hölder, $|\varphi_h * g|^p(x) \leq \left(\int_{\mathbb{R}^n} \varphi(z) dz\right)^{p/q} = \int \varphi(z) |g(x - zh)|^p dz$.

How we integrate over x and use Fubini to interchange the order of integration, and it follows.

Topics in L^2

Fourier Transform: $S^1 = [-\pi, \pi]/\sim$ and $f \in C^2(S^1)$, then $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ixn} dx$, and $f(x) \sim \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{ixn}$. So $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$.

8 Lecture 8

Today we will talk about the Fourier Transform on $\mathbb{R}^n$.

The inspiration is that functions on S^1 can be written as $f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx}$ where $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$.

The e^{ixu} are eigenfunctions for the laplacian on S^1.

What is the analogue for $\mathbb{R}^n$?

Let $f(x) \in L^1(\mathbb{R}^n)$ and e^{ixu} corresponds to $e^{ix \cdot \xi}$ with $\xi \in \mathbb{R}^n$. So define $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$.

Lemma 8.1. *$\mathscr{F}(f) = \hat{f}$ is a continuous map $L^1(\mathbb{R}^n) \to C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ equipped with metric $\|f - g\| = \|f - g\|_{L^\infty}$.*

Proof. Continuity is simple, we show that $\mathscr{F} : L^1 \to C^0$ and $\mathscr{F} : L^1 \to L^\infty$.

$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^n} |f(x)| dx < \infty$, $\hat{f}(\xi) \in L^\infty$.

For continuity, $\hat{f}(\xi + h) - \hat{f}(\xi) = \int (e^{-2\pi i x(\xi + h)} - e^{-2\pi i x \cdot \xi}) f(x) dx$. The dominated convergence theorem $|e^{-2\pi i x(\xi + h)} - e^{2\pi i x \cdot \xi}| \leq 2$, $\lim_{h \to 0} e^{-2\pi i x(\xi + h)} - e^{-2\pi i x \cdot \xi} = 0$ pointwise, and so by Lebesgue Dominated COnvergence, the limit of $|\hat{(}f)(\xi + h) - \hat{f}(\xi)| \to 0$. $\square$

Question: Is the map surjective?

Lemma 8.2 (Riemann-Lebesgue Lemma)**.** *Let $f \in L^1(\mathbb{R}^n)$, then $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$.*

Proof. (Density Argument)

Last time, we showede that $C_c^\infty \subset L^1(\mathbb{R}^n)$ is dense.

For all $\epsilon > 0$, there exists $g \in C_c^\infty(\mathbb{R}^n)$ such that $\|f - g\|_{L^1} < \epsilon$. This implies that $\|\hat{f} - \hat{g}\|_{L^\infty} < \epsilon$.

It suffices to show that $\lim_{|\xi|\to\infty} \hat{g}(\xi) = 0$. Assume that $|\xi|$ is very large. Some $|\xi_k| > |\xi|/\sqrt{n}$. Pick such a ξ_k. Then $|\hat{g}(\xi)| = |\int g(x)e^{-2\pi ix\cdot\xi}dx| = |\int \left(\frac{1}{2\pi i\xi_k}\frac{\partial}{\partial x_k}\right)^N g(x)e^{-2\pi ix\cdot\xi}dx| \leq C_N/|\xi_k|^N \leq \tilde{C}_N/|\xi|^N$.

In particular, $\hat{g}(\xi)$ vanishes rapidly at infinity. Therefore $\limsup_{|\xi|\to\infty} |\hat{f}(\xi)| < \epsilon$, and is zero as ϵ was arbitrary. □

Remark: Can $\hat{g}(\xi)$ be compactly supported? $\int_{\mathbb{R}^n} e^{-2\pi ix\cdot\xi}g(x)dx$.

The answer is no (this corresponds to the Heisenberg Uncertainty Principle)

A proof of this is given by generalizing to transforming to complex valued functions, and this gives a holomorphic function, which cannot be of compact support.

Big Theorems: The Fourier Inversion Theorem, Plancherel's Theorem (which allows the extension of $\mathscr{F}$ to L^2)

Schwartz class of functions: $S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) | \sup_{x\in\mathbb{R}^n} |x|^\alpha \left|\frac{\partial}{\partial x^\beta}f(x)\right| < \infty$ for all $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}^k$.

Lemma 8.3. *$\mathscr{F}$ maps $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$.*

Proof. $\hat{f}(\xi) = \int f(x)e^{-2\pi ix\cdot\xi}dx$. Then $\frac{\partial}{\partial\xi^\alpha}\hat{f}(\xi) = (-2\pi i)^{|\alpha|}\int\prod_{\ell=1}^n x_\ell^{\alpha_\ell}f(x)e^{-2\pi ix\cdot\xi}dx$.

If $\beta \in \mathbb{N}^n$, then $\xi^\beta\hat{f}(\xi) = (-2\pi i)^{|\alpha|-|\beta|}\int x^{n+1}\frac{\partial^\beta}{\partial x^\beta}\prod x_\ell^{\alpha_\ell}f(x)e^{-2\pi ix\cdot\xi}dx$.

Hence $\sup_{\xi\in\mathbb{R}^n}|\xi^\beta\frac{\partial}{\partial\xi^\alpha}\hat{f}(\xi)| \leq C_{n,\alpha,\beta}\sup_{x\in\mathbb{R}^n}$ obvious thing. □

Note that by tdensity of $C_c^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ we have that $S(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is dense.

We establish Fourier Inversion and Plancherel for S and pass to L^2 by density.

Lemma 8.4 (easy)**.** *Let $f, g \in S(\mathbb{R}^n)$, then $|int\hat{f}(\xi)g(\xi)d\xi = \int f(x)\hat{g}(x)dx$.*

Fubini. $\int_{\mathbb{R}^n}\hat{f}(\xi)g(\xi)d\xi = \int\int f(x)e^{-2\pi ix\cdot\xi}dxg(\xi)d\xi$.

This is then equal to $\int\int e^{-2\pi ix\cdot\xi}g(\xi)d\xi f(x)dx = \int f(x)\hat{g}(x)dx$. □

Theorem 8.5 (Fourier Inversion)**.** *Let $\check{f}(x) = \int f(\xi)e^{2\pi ix\cdot\xi}d\xi$. Then if $f \in S(\mathbb{R}^n)$, we have $(\check{\hat{f}}) = f$ and $(\hat{\check{f}}) = f$.*

Proof. $\check{\hat{f}} = \int\int f(y)e^{-2\pi iy\cdot\xi}dye^{2\pi ix\cdot\xi}d\xi$. A physicist would just say $\int f(x)\int e^{2\pi i(x-y)\cdot\xi}d\xi dy = \int\delta(x-y)f(y)dy = f(x)$ with $\int e^{ix\xi}dx = \delta(\xi)$.

We, however, must justify this better.

The actual proof replaces $e^{i(x-y)\cdot\xi}$ with a dampened version. Introduct $\check{(f)}_\epsilon = \int f(\xi)e^{2\pi ix\cdot\xi-\epsilon|\xi|^2}d\xi$ for $\epsilon > 0$.

Then $(\check{\hat{f}})_\epsilon = \int\hat{f}(\xi)e^{2\pi ix\cdot\xi-\epsilon|\xi|^2}d\xi = \int\int f(y)e^{-2\pi iy\xi}dye^{2\pi ix\xi-\epsilon|\xi|^2}d\xi$. By Fubini, this gives $\int f(y)\int e^{2\pi i(x-y)\xi-\epsilon|\xi|^2}d\xi dy$, we take $K_\epsilon(x-y) = \int e^{-\left(\sqrt{\epsilon}\xi-\pi i(x-y)/\sqrt{\epsilon}\right)^2}e^{-\pi^2(x-y)^2/\epsilon^2}d\xi$.

??? □

Definition 8.1. *A family of functions $K_\epsilon(x)$ indexed by $\epsilon > 0$ which are positive, integrate to 1, and* $\lim) \epsilon \to 0 \int_{M>\delta} K_\epsilon(x)dx = 0$ *for any $\delta > 0$ is called an approximate identity.*

Lemma 8.6. *Assume K_ϵ is an approximate identity. Then $K_\epsilon * f \to f$ if $f \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ pointwise and uniformly on compact subsets of $\mathbb{R}^n$.*

Proof. $K_\epsilon * f - f)(x) = \int K_\epsilon(x-y)f(y)dy - f(x) = \int K_\epsilon(x)[f(x-y) - f(x)]dy$. Given $\mu > 0$, choose $\delta > 0$ such that $|f(x-y) - f(x)| < \mu/2$ whenever $|y| < \delta$.

Then choose $\epsilon > 0$ such that $\int_{|y|\geq\delta} K_\epsilon(y)dy < \mu/2\|f\|_{L^\infty}$. Then $|K_\epsilon * f - f| \leq \left|\int_{|y|<\delta} K_\epsilon(y)[f(x-y) - f(x)]dy\right| + \left|\int_{|y|\geq\delta} K_\epsilon(y)[f(x-y) - f(x)]dy\right| < \mu$. □

Completing the Fourier Inversion Proof:

$(\check{\hat{f}}) = \int_{\mathbb{R}^n} f(y)K_\epsilon(x-y)dy \to f(x)$ by the lemma.

This is the same as $\int \hat{f}(\xi)e^{2\pi i x\cdot\xi - \epsilon|\xi|^2}d\xi \to \int \hat{f}(x)e^{2\pi i x\cdot\xi}d\xi = (\check{\hat{f}})(x)$.

Theorem 8.7 (Plancherel's Theorem)**.** *If $f, g \in S(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} \hat{f}(\xi)\bar{\hat{g}}(\xi)d\xi = \int_{\mathbb{R}^n} f(x)\bar{g}(x)dx$.*

In particular, if $f = g$, then $\int |\hat{f}(\xi)|^2 d\xi = \int |f(x)^2|dx$.

Proof. $\bar{\hat{g}}(\xi) = \check{\bar{g}}(\xi)$. Assume that $\bar{G} = \hat{h}(\xi)$, then $\check{\bar{g}} = h$.

$$\int \hat{f}(\xi)\bar{\hat{g}}(\xi)d\xi = \int \hat{f}(\xi)h(\xi)d\xi = \int f(x)\hat{h}(x)dx = \int f(x)\bar{g}(x)dx$$

□

Now we can extend $\mathscr{F}$ to a map $L^2 \to L^2$. Given $f \in L^2$, by the density of $S(\mathbb{R}^n)$ in L^2, we find $\{f_n\}_{n\geq 1}$ such that $f_n \to f$ in L^2 sense.

$\hat{f}_n \to g$ by Plancherel and completeness of L^2, so g is unique (up to a set of measure zero), and so we define $\hat{f} = g$.

9 Lecture 9

Question: Can $\mathscr{F}$ be defined for $f \in L^p$ for $p \neq 1, 2$?

The answer is yes by general principle that $T : L^{p_0} \to L^{q_0}$, $L^{p_1} \to L^{q_1}$ can by interpolated to $T : L^p \to L^q$ for $0 \leq \theta \leq 1$ and $1/p = \theta/p_0 + (1-\theta)/p_!$ and q satisfying the same.

A consequence is the following:

Theorem 9.1 (Young's Inequality)**.** *$f \in L^p$ and $1 \leq p \leq 2$, then $\hat{f}$ can be defined in L^q for $1/p + 1/q = 1$ and $\|\hat{f}\|_{L^q} \leq \|f\|_{L^p}$.*

An application of the Fourier Tranform:

The Linear Schodinger EQuation is $(i\partial_t + \Delta)u(t,x) = 0$. The Cauchy problem is: if $u(0,x) = f(x)$ in $S(\mathbb{R}^n)$ find $u(t,x)$.

Then $u(t,x) = \int_{\mathbb{R}^n} \hat{u}(t,\xi) e^{2\pi i x\cdot\xi} d\xi$ by Fourier inversion, and $(i\partial_t + \Delta)u(t,x) = \int_{\mathbb{R}^n} (i\partial_t - 4\pi|\xi|^2)\hat{u}(t,\xi)e^{2\pi i x\cdot\xi} d\xi = 0$.

So $\hat{u}(t,\xi) = e^{-4\pi i t|\xi|^2}\hat{u}(0,\xi) = e^{-4\pi i t|\xi|^2} f(\xi)$. So then $u(t,x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} e^{-4\pi i t|\xi|^2} f(\xi) d\xi$. By $\mathscr{F}$ is an isometry of L^2, we have that $\|u(t,x)\|_{L^2} = \|f\|_{L^2}$.

General Banach Space Theory

First we will look at Fréchet Spaces.

Definition 9.1 (Banach Space). *A Banach Space is a vector space over $\mathbb{R}$ or $\mathbb{C}$ equipped with a norm $|-| : V \to \mathbb{R}_{\geq 0}$ such that $|x+y| \leq |x| + |y|$, $|xy| = |x||y|$ and $|x| = 0$ iff $x = 0$.*

Further, equipping V with distance $d(x,y) = |x-y|$ makes it a complete metric space.

Definition 9.2 (Frechet Space). *Let V be a vector space with a metric d which is translation invariant ($d(x,y) = d(x-y,0)$), and such that scalar multiplication is continuous and such that (V,d) is complete.*

Definition 9.3 (Bounded). *Let X be a Frechet space. A subset $U \subset X$ is bounded if for every open set $0 \in V \subset X$ there is an $\epsilon > 0$ such that $\forall \alpha \in F$, $|a| < \epsilon$ we have $\alpha U \subset V$*

For a Banach Space, this coincides with the usual notion.

Theorem 9.2 (Principle of Uniform Boundedness). *Let $\{T_a | a \in A\}$ collection of continuous linear maps $X \to Y$ with X, Y Frechet Spaces. Then if $\forall x \in X$, the set $\{T_a x | a \in A\}$ is bounded, then the family is uniformly continuous (equicontinuous according to Rudin).*

Proof. We will use the Baire Category Theorem, in the form that if $X = \cup A_n$ where A_n is closed, then at least one A_n has nonempty interior.

Let $|y| = d(y,0)$, we apply the Baire Category Theorem to the following closed sets: $A_k = \{x \in X | \frac{1}{k}|T_a(x)| + |\frac{1}{k}T_a(-x)| \leq \epsilon/2 \forall a \in A\}$.

By the continuity of T_a, these are closed, and so we must check that $X = \cup A_k$. For all $x \in X$, $\sup_{a\in A} |T_a x| = k(x) < \infty$. Thus, there exists $\epsilon > 0$ such that forall $|\alpha| < \epsilon$, $|\alpha T_a x| < 1$, and so the union works.

So now Baire implies that for some A_ℓ, it contains a nonempty interval. And so $|1/\ell T_a(x_0 + x)| \leq \epsilon/2$ for $|x| < \delta$, the length of this interval. Then $|1/\ell T_a(x)| \leq |1/\ell T_a(x_0+x)| + |1/\ell T_a(-x_0)| \leq \epsilon$ for all $|x| < \delta$.

And so the map $x \to x/\ell$ for $\ell \geq 1$ is a homeomorphism and hence there exists $\delta_0(\delta,\ell)$ such that $|x| < \delta_0 \Rightarrow |T_a(x)| < \epsilon$ for all $a \in A$. □

Next: Equivalence of continuity and boundedness.

Theorem 9.3. *Let $T : X \to Y$ be a linear map between frechet spaces. Then T is bounded iff T is continuous.*

What we mean is that T is bounded iff $T(U)$ bounded whenever U is bounded. So if X, Y are Banach Spaces, then T bounded iff $\sup_{|x|\leq 1} |Tx| < \infty$.

Proof. Only if: T cont implies T bounded. Let $U \subset X$ be a bounded set and $0 \in V \subset Y$ an open set. $\alpha T(U) \subset V$ if $|\alpha|$ small enough. By continuity of T, there exists $0 \in \tilde{V} \subset X$ open such that $T(\tilde{V}) \subset V$. By the boundedness of U, there exists $\epsilon > 0$ such that $\alpha U \subset \tilde{V}$ for all a with $|a| < \epsilon$. This implies that $T(\alpha U) = \alpha T(U) \subset V$ and so $T(U)$ is bounded.

If: T bounded implies T continuous. Need to show that if $x_i \to 0$ then $Tx_i \to 0$. But the $|x_i| \to 0$...(zoned out).

Claim: $U\{k_i x_i\}_{i\geq 1}$ is bounded. $\sup |k_i x_i| < \infty$ and $\sup |T(k_i x_i)| < \infty$, and so $T(x_i) = \frac{1}{k_i} T(k_i x_i)$. So because $\{T(k_i x_i)\}$ is bounded, for all $V \ni 0$, there exists $\epsilon > 0$ such that $\alpha\{T(k_i x_i)\} \subset V$ if $|\alpha| < \epsilon$.

We now pause to note that a compact subset of a Frechet Space is a bounded set.

There exists $\delta > 0$, $0 ni U \subset X$ such that $\beta U \subset V$ for all $|\beta| < \delta$.

Also, we have $\cup nU = X$, and by compactness of B, $B \subset \cup nU$, hence if $\epsilon = \delta/m$, then $\alpha B \subset \cup \alpha n U \subset V$ for all α with $|\alpha| < \epsilon$. □

10 Lecture 10

There are three important theorems: The Open Mapping Theorem, the Closed Graph Theorem and the Hahn-Banach Theorem.

Open Mapping Theorem

Motivation: If $T : V \to W$ is a linear map of finite dimensional vector spaces, then there exists a basis e_i for V, f_j for W with $T(e_i) = f_i$ and $T(e_n) = 0$ for e_n not corresponding to f_j.

In particular, if we equip V with a norm, and U an open neighborhood of the origin, then $T(U)$ is open.

Theorem 10.1 (Banach's Open Mapping Theorem)**.** *Let X, Y be Frechet Spaces (eg Banach Spaces) and T a continuous surjection. Then T is open.*

Corollary 10.2. *If $T : X \to Y$ is a continuous linear bijection at the set theoretic level, then T is continuously invertible.*

We prove the open mapping theorem:

Proof. Step 1: Let $0 \in G \subset X$ be open. Then $\bar{TG}$ contains an open neighborhood of $0 \in Y$. This is by the Baire category theorem plus a trick to trnalate back to the origin. Choose an open neighborhood $0 \in M \subset X$ such that the difference set is contained in G. Consequence of continuity of addition and scalar multiplication is that $X = \cup_{n=1}^{\infty} nM$, that is, given $x \in X$, $0x = 0$ and by continuty of scalar mult there exists $\epsilon > 0$ such that $\forall |\alpha| < \epsilon$, $\alpha x \in M$. Then choose $n > 1/\epsilon$, $x \in nM$.

Now, by the surjectivity of T, $Y = \cup nTM = \cup n\bar{T}M$. By the Baire category theorem, one of the $n\bar{T}M \supset V$ an open set which is nonempty.

We conclude step 1 by noticing that $\bar{TG} \supset TM \bar{-} TM = \bar{TM} - \bar{TM} \subset \frac{1}{n}V - \frac{1}{n}V \ni 0$.

Step 2: Now, we must show that TG contains an open neighborhood of 0, $B_{\epsilon_0}(0) = \{x \in X | d(x, 0) < \epsilon_0\}$.

By the first step, $\overline{T(B_{\epsilon_1}(0))} \supset B_{\eta_0}(0)$ if η_0 is sufficiently small.

Given $y \in B_{\eta_0}(0)$, there exists $x_1 \in B_{\epsilon_0}(0)$ such that $d(y, Tx_1) < \epsilon_0/2$.

By linearity, we also have $T(B_{\epsilon_0/2}(0)) \supset B_{\eta_0/2}(0)$. Hence there is $x_2 \in B_{\epsilon_0/2}(0)$ such that $d(y - Tx_1, Tx_2) < \eta_0/4$ iff $d(y - Tx_1 - Tx_2, 0) < \eta_0/4$.

PRoceeding inductively, we obtain a sequence of corrections $x_k \in B_{\epsilon_0/2^k}(0)$ such that $d(y - T(\sum x_k), 0) < \eta_0/2^{k+1}$.

By the triangle inequality, $z_k = \sum x_n$ is Cauchy, and by the completeness of X, there exists $x \in B_{2\epsilon_0}(0)$ such that $z_k \to x$. By the continuity of T, $Tx = y$.

For general open $0 \in G \subset X$, find $B_{2\epsilon_0}(0) \subset G$.

Step 3: Now, given $G \subset X$ open, $x \in G$, let $\epsilon_0 > 0$ such that $x + B_{2\epsilon_0}(0) \subset G$. Then $TG \supset Tx + T(B_{2\epsilon_0}(0)) \supset Tx + B_{\eta_0}(0)$. □

Corollary 10.3 (Closed Graph Theorem)**.** *Let X, Y be Frechet Spaces and $T : X \to Y$ a linear map. Also, let $\Gamma_T = \{(x, y) \in X \times Y | y = Tx\}$ be the graph of T. Then T is continuous iff Γ_T is closed.*

Proof. (only if): Assume that T is continuous. Consider a sequence $\{(x_k, Tx_k)\} \subset \Gamma_T$. Assume that this sequence converges to (x, y). By continuity, $x_k \to x$ implies $Tx_k \to Tx$, so $(x, y) \in \Gamma_T$.

(if): Now assume that Γ_T is closed. Then Γ_T is a vector subspace of $X \times Y$, and so by closedness, it's a Frechet space. Look at the projection maps π_X, π_Y. Both maps are continuous and π_X is surjective and invertible at the set theoretic level, and so π_X^{-1} is continuous by the open mapping theorem. And so $T = \pi_Y \circ \pi_X^{-1}$, which is continuous. □

Hahn-Banach Theorem

This is a machine to generate lots of continuous linear maps.

Definition 10.1 (Normed Linear Space)**.** *A normed linear space is defined like a Banach space, but without the requirement of completeness.*

Observation: A subset $U \subset X$ of a normed linear space is bounded in the sense of Frechet Spaces iff $\sup_{x \in U} |x| < \infty$. Alinear map $T : X \to Y$ between normed linear spaces is bounded iff $\sup_{x \in X, |x| \leq 1} |Tx| < \infty$.

Definition 10.2 ($B(X, Y)$)**.** *Let X, Y be normed linear spaces (ie, F-spaces or B-spaces) then $B(X, Y)$ denotes the set of all continuous linear maps $X \to Y$.*

If $Y = \mathbb{R}$ or $\mathbb{C}$, the underlying field, then $B(X, Y) = X^*$ is called the dual space. Furthermor, $B(X, Y)$ is equipped with a norm.

Define $|T| = \sup_{|x| \leq 1} |Tx|$ provided that X, Y normed.

Lemma 10.4. *Assume that X, Y are normed and Y is complete. Then so is $B(X, Y)$.*

Proof. Let $\{T_n\}_{n\geq 1}$ be a Cauchy sequence in $B(X,Y)$. Then $\forall x \in X$, $\{T_n x\} \subset Y$ is Cauchy, and since $|T_n x - T_m x| \leq |(T_n - T_m)x| \to 0$, then $\lim_{n\to\infty} T_n x$ exists for all $x \in X$. The map $T : x \to \lim_{n\to\infty} T_n x$ is linear and bounded, since $\lim_{n\to\infty} |T_n x| \leq \liminf_{n\to\infty} |T_n||x| = C|x|$. □

In particular, the dual of a Banach space is a Banach Space.

Theorem 10.5 (Hahn-Banach Theorem)**.** *Let X be an $\mathbb{R}$-vector space and $p : X \to \mathbb{R}$ a function satisfying $p(x+y) \leq p(x) + p(y)$ and $p(\alpha x) = \alpha p(x)$ for $\alpha \geq 0$. Further, let $Y \subset X$ be a subspace and $f : Y \to \mathbb{R}$ a linear map with $f(x) \leq p(x)$ for all $x \in Y$. Then there exists a real linear map $F : X \to \mathbb{R}$ such that $F|_Y = f$ and $f(x) \leq p(x)$ for all $x \in X$.*

Proof. We implement Zorn's Lemma. Let $S = \{(g, \tilde{Y}) | \tilde{Y} \supset Y, g : \tilde{Y} \to \mathbb{R}$ and $g|_Y = f$ and $g(x) \leq p(x) \forall x \in \tilde{Y}\}$.

We say that $h \geq g$ if h extends g. Claim: This satisfies the property of the statement of Zorn's Lemma

Let $E \subset S$ be totally ordered. Define $g_E : \cup_{n\in E} \tilde{Y}_n \to \mathbb{R}$ by $g_E|_{\tilde{Y}_n} = h$. This is consistant by the total orderedness of E. Then g_E is an upper bound.

By Zorn, we choose a maximal element, $g \in S$. So now we simply need to show that g is defined on all of X. We argue by contradiction. Assume that $Y \subset \tilde{Y}_g \subsetneq X$. Choose $y_1 \in X \setminus \tilde{Y}_g$. Define $\tilde{Y} = \operatorname{span}\{y_1, \tilde{Y}_g\} \supsetneq \tilde{Y}_g$, if $y \in \tilde{Y}$, $y = \alpha y_1 + y_0$, $\alpha \in \mathbb{R}$ and $y_0 \in \tilde{Y}_g$. Try an ansatz, $g_1(y) = c\alpha + g(y_0)$ for some $c \in \mathbb{R}$.

Question: Can we arrange that $g_1(y) \leq p(y)$?

Yes, by algebraic trickery: $y_0, \tilde{y}_0 \in \tilde{Y}_g$, $g(\tilde{y}_0) - g(y_0) = g(\tilde{y}_0 - y_0) \leq p(\tilde{y}_0 - y_0) \leq p(\tilde{y}_0 + y_1) + p(-y_0 - y_1)$, which implies that $-p(-y_0 - y_1) - g(t_0) \leq p(\tilde{y}_0 + y_1) - g(\tilde{y}_0)$.

So we have $-\infty \leq \sup_{y_0 \in \tilde{Y}_g} -p(-y_0 - y_1) - g(y_0) \leq \inf_{\tilde{y}_0 \in \tilde{Y}_g} p(\tilde{y}_0 + y_1) - g(\tilde{y}_0) < \infty$, and take c to be in the middle.

So now $p(y + y_1) - g(y) \geq 0$ and $-p(-y - y_0) - g(y) \leq c$ for all $y \in \tilde{Y}_g$.

Now, for $y_2 = y_0 + \alpha y_1 \in \tilde{Y}$, $g(y_2) = g(y_0) + \alpha c \leq p(y_0 + \alpha y_1)$?

This is true if $\alpha = 0$, $\alpha > 0$, $p(y_0 + \alpha y_1) = \alpha p(y_0/\alpha + y_1) \geq \alpha[c + g(y_0/\alpha)] = \alpha c + g(y_0)$ or $\alpha < 0$, $p(y_0 + \alpha y_1) = |\alpha| p(y_0/|\alpha| - y_1) \geq -|\alpha|(g(-y_0/\alpha) + c) = g(y_0) + c\alpha$. □

11 Lecture 11

Today we will duscuss applications of Hahn-Banach, in particular, reflexivity of Banach Spaces.

Theorem 11.1. *Let $Y \subset X$ a normd linear space over $\mathbb{R}$ or $\mathbb{C}$. Let $y^* \in Y^*$, then there exists $x^* \in X^*$ such that $x^*|_Y = y^*$ and $|x^*| = |y^*|$.*

Proof. If we are over $\mathbb{R}$, then we define $p(x) = |x||y^*|$, and $y^*(x) = p(X)$. Then for all $x \in Y$ by Hahn-Banach???? Stupid eraser, I hate this class.

Now we assume that X is a $\mathbb{C}$-vector space. Then we write $x^*(y) = f_1(y) + if_2(y)$ for $f_i : Y \to \mathbb{R}$. Then the f_i are real linaer and so $|f_1(y)| \leq |x^*(y)| \leq |y^*||y|$, and we apply Hahn-Banach to $f_1(y)$.

This gives us $F_1 : X \to \mathbb{R}$ and $F_1|_Y = f_1$. So $|F_1(x)| \leq |y^*||x|$. Now we define $x^*(x) = F_1(x) - iF_1(ix)$ which is, a priori, just $\mathbb{R}$-linear. But actually it is $\mathbb{C}$-linear, we just need to check that $x^*(ix) = ix^*(x)$, which holds because $x^*(ix) = F_1(ix) - iF_1(-x) = iF_1(x) + f_1(ix) = i(F_1(x) - iF_1(ix))$.

Further, we claim that $x^*|_Y = y^* = f_1(y) + if_2(y)$, this is because $y^*(iy) = f_1(iy) + if_2(iy) = iy_*(y) = if_1(y) - f_2(y)$ and so $f_2(y) = -f_1(iy)$ and $f_*(y) = f_1(y) - if_1(iy)$ for all $y \in Y$, and so $F_1(x) - iF_1(ix)|_Y = y^*$.

We now only need to check that $|x^*| \leq |y^*|$ for arbitrary $x \in X$. Write $x^*(x) = re^{i\theta}$ for $r > 0$ and $\theta \in \mathbb{R}$. Then $|x^*(x)| = x^*(e^{-i\theta}x) = F_1(e^{-i\theta}x) \leq |y^*||e^{-i\theta}x| = |y^*||x|$. □

Consequences of this theorem: One can use leinear functionals (continuous) to separate points, or points and closed subspaces.

Lemma 11.2. *Let $Y \subset X$ normed linear over $\mathbb{R}$ or $\mathbb{C}$ and $x \in X$ satisfies ???*

Proof. We construct x^* first on linear span of x and Y and then extend it via the preceding theorem to all of X. Let $Z = \text{span}\{x, Y\}$. So if $z \in Z$, $z = \alpha x + y$ for a unique $\alpha \in F$ and $y \in Y$. Then define $x^*(z) = \alpha$.

We need to check that $x^* : Z \to \mathbb{R}$ or $\mathbb{C}$ satisfies teh necessary bound $|x^*| = 1/d$. $|z| = |y + \alpha x| = |\alpha||y/\alpha + x| \geq |\alpha d|$ by the definition of d for all $\alpha \neq 0$. Thus, $|x^*(z)| \leq |z|/d \Rightarrow |x^*| \leq 1/d$.

To see that $|x^*| \geq 1/d$, choose $y_n \in Y$ such that $|x - y_n| \to d$ as $n \to \infty$. Then $x^*(x - y_n) \leq |x^*||x - y_n| \to |x^*|d$ and $x^*(x - y_n) = 1$, so $1 \leq |x^*|d$. □

Corollary 11.3. *Let $Y \subset X$ a closed linear subspace, X normed. Let $x \in X \backslash Y$, then there exists $x^* : X \to F$ such that $x^*(x) = 1$ and $x^*|_Y = 0$.*

Proof. Note that $\int_{y \in Y} |x - y| > 0$. □

Corollary 11.4. *$\forall x \in X, x \neq 0, \exists x^* \in X^*$ with $|x^*| = 1$ and $x^*(x) = |x|$.*

Proof. Simply use $Y = \{0\}$. □

Alternative statement of this: If $x_1 \neq x_2 \in X$, then there is a functional $x^* \in X^*$ such that $x^*(x_1) \neq x^*(x_2)$.

That is, there are enough continuous linear functions to separate points.

Corollary 11.5. *Let $x \in X$ a normed linear space. Then $|x| = \sup_{x^* \in X^*, |x^*| \leq 1} |x^*(x)|$.*

The last corollary is important, because of the following relation: $(X^*)^* = X^{**}$.

There is a canonical map $X \to X^{**}$ giveb by $x \in X$ maps to $\hat{x} = \kappa(x) \in X^{**}$. Then $\hat{x}(y^*) = y^*(x)$ for an arbitrary element $y^* \in X^*$.

This is an element of X^{**} because $|\hat{x}(y^*)| \leq |y^*||x|$, $\hat{x} : X^* \to F$ is bounded and linear. On account of the preceding corollary, $|\hat{x}| = |x|$, as $|\hat{x}| = \sup_{y^* \in X^*, |y^*| \leq 1} |y^*(x)| = |x|$.

This says that the canonical map $\kappa : X \to X^{**}$ is an isometry onto a subspace of X^{**}.

Issue: When is this map onto? When it is, $X \simeq X^{**}$. This is only possibly if X is a Banach Space!

Important examples where $X \simeq X^{**}$: If X is a Hilbert space, say $L^p(X)$ for $1 \leq p \leq \infty$ are examples that are "reflexive."

Definition 11.1 (Reflexive)**.** *Let X be a Banach space. If κ is surjective, then X is called reflexive.*

Why is it important? Reflexive X has certain weak completeness and for bounded subsets of X, weak compactness properties.

Definition 11.2 (Separable)**.** *Let (X, d) be a metric space. It is called separable if there exists a countable dense subset.*

Definition 11.3 (Weakly Convergent)**.** *Let X be a normed linear space or a Frechet space. A sequence $\{x_n\} \subset X$ is called weakly convergent provided that there exists $x \in X$ such that $x^*(x) = \lim_{n\to\infty} x^*(x_n)$ for all $x^* \in X^*$.*

If $X = L^2(\mathbb{R})$, then look at a travelling compactly supported wave. Let f_n be the wave front starting at n. This converges weakly to zero, but is not convergent.

If x_n is weakly convergent, and x satisfying the above is called a weak limit. A subset $A \subset X$ is called weakly sequentially compact provided that each sequence $x_n \in A$ has a weakly convergent subseuqnece. A sequence is called weakly Cauchy proved that $\{x^*(x_n)\} \in F$ is a Cauchy sequence for all $x^* \in X^*$.

Theorem 11.6 (Main Theorem on Reflexive Banach Spaces)**.** *A reflexive Banach space is weakly complete. A subset of a reflexive Banach Space is weakly sequentially compact iff it is bounded.*

We will prove this in a sequence of steps.

Lemma 11.7. *A weakly convergent sequence in a normal linaer space has a unique limit.*

Proof. Assume there are two. Contradiction with the point separation property. ☐

Lemma 11.8. *Let $\{x_n\}_{n\geq 1}$ a sequence in X a normed reflexive space. Then if* $\sup |x^*(x_n)| < \infty$ *for all $x^* \in X^*$, then* $\sup_n |x_n| < \infty$.

Proof. Consider $\hat{x}_)n \subset X^{**}$. By assumption, $\sup_n |\hat{x}_n(x^*)| < \infty$ for all $x^* \in X^*$. By the principle of uniform boundedness, there exists $\delta > 0$ such taht $|\hat{x}_n(x^*)| < 1$ if $|x^*| < \delta$.

Thus, $|\hat{x}_n| < 1/\delta$ and since the embedding is isomoetric, $|x_n| < 1/\delta$ ☐

Lemma 11.9. *A weakly convergent sequence $\{x_n\}_{n\geq 1}$ in a normed linear space is bounded, it's limit x is in the closure of the linear span of the x_i and $|x| \leq$* $\liminf |x_n|$.

Proof. Boundedness follows from preceding lemma. Assertion about the linear span follows from Hahn-Banach. For final inequality, we have $|x^*(x)| = \lim_{n\to\infty} |x^*(x_n)| \leq \liminf |x^*||x_n|$. Then we use $|x| = \sup_{|x^*|<1} |x^*(x)|$. $\square$

Lemma 11.10. *If the dual X^* of a normed linear space is separable, then so is X.*

Proof. Let $\{x_n^*\}_{n\geq 1} \subset X^*$ be a countable dense set and choose $x_n \in X$ such that $|x_n| \leq 1$ and $|x_n^*(x_n)| \geq |x_n^*|/2$.

Claim: The set of finite linear combinations L of the x_n with rational coefficients is dense in X.

If not, then $\bar{L} \neq X$, by one of the collorlaries of Hahn-Banach, there exists $x^* \in X^* \setminus \{0\}$ such that $x^*(L) =?$. BY density, let $x_{n_i}^* \to x^*$, then $|x^* - x_{n_i}^*| \geq |(x^* - x_{n_i})(x_{n_i})| = |x_{n_i}(x_{n_i})| \geq |x_{n_i}^*|/2$.

Since $|x^* - x_{n_i}^*| \to 0$, we have that $|x_{n_i}^*| \to 0$,a nd so $x^* = 0$. $\square$

12 Lecture 12

Reflexivity: Main Theorem: If X is a reflexive Banach Space, then $A \subset X$ is weakly sequentially compact iff bounded. Here A needn't be a subspace.

Last time we proved Lemma -1 which said that a weakly convergent sequence has a unique limit and lemma 0 which says that if $\{x_n\}_{n\geq 1} \subset X$ a normed linear space. Assume that $\sup_n |x^*(x_n)| < \infty$ for all $x^* \in X^*$, then $\sup_n |x_n| < \infty$. In particular, if $\{x_n\}$ is weakly convergent or weakly cauchy, then $\sup_n |x_n| < \infty$.

Lemma 12.1. *The limit x of a weakly convergent sequence $\{x_n\}$ is in the closure of the linear space of the x_n and $|x| \leq \limsup_n |x_n|$.*

Lemma 12.2. *If the dual X^* of a normed linear space is separable, so is X.*

We will now prove the main theorem:

Theorem 12.3. *If X is a reflexive Banach Space, then $A \subset X$ is weakly sequentially compact iff bounded.*

Strategy for the if part(hard): Given $\{y_n\} \subset A$ we want a subsequence $\{y_{n_k}\}$ such that $y^*(y_{n_k})$ converges for all $y^* \in X^*$. As $(Y^*)^* = Y^{**} = Y$ =closure of the span of the y_i's.

Lemma 12.4. *Let X, Y be Banach spaces and $T_n : X \to Y$ bounded linear operators. Then $\lim_{n\to\infty} T_n x = Tx$ exists and defines a continuous linear map iff*

1. *The limit exists for a fundamental set: ie, one whose linear span is dense in X.*
2. *$\forall x \in X$, $\sup_n |T_n x| < \infty$.*

Proof. Only if is clear. For if, assume $T_n x$ converges for $D \subset X$ fundamental. NBy the principle of uniform boundedness, given $\epsilon > 0 \exists \delta > 0$ such that $|T_n x| < \epsilon$ for all $x, |x|, \delta$ and all n.

Given $x \in X$, choose $y \in \text{span}(D)$ such that $|x - y| < \delta$. Further choose $n_0 = n_0(\epsilon, y)$ such that $\forall n, m \geq n_0(\epsilon, y)$, $|T_n y - T_m y| < \epsilon$.

Thus, $|T_n(x) - T_m(x)| \leq |T_n(x) - T_n(y)| + |T_n(y) - T_m(y)| + |T_m(y) - T_m(x)| \leq 3\epsilon$ by our choices. Hence the sequence $T_n x$ is Cauchy for every $x \in X$.

???

Hence $T = \lim_{n\to\infty} T_n : X \to Y$ is continuous. □

Lemma 12.5 (Inheritance of Reflexivity by Closed Subspaces)**.** *A closed subspace Y of a reflexive Banach space X is also reflexive.*

Proof. Let $X^* \subset Y^*$. Then $r : X^* \to Y^*$ gives rise to $r^* : (Y^*)^* \to X^{**}$. To define it, given $y^{**} \in Y^{**}$, ???

So we have $\kappa : X \to X^{**}$ an isometric embedding canonically. So $\kappa(x)(x^*) = x^*(x)$ for all $x^* \in X^*$. Assume that we have shown that (N) $\kappa^{-1}(r^*(Y^{**})) \subset Y$. Let's conclude the proof from (N). Given $y^{**} \in Y^{**}$. Then $r^*(y^{**}) \in X^{**}$. Further, given an arbitrary element of $y^* \in Y^*$, we can choose an extension $x^* \in X^*$ such that $r(x^*) = y^*$. THis is possible by a corollary of Hahn-Banach.

Now we have $y^{**}(y^*) = y^{**}(r(x^*)) = r^*(y^{**})(x^*) = x^{**}(x^*) = \kappa(x)(x^*)$ by the reflexivity of X. This is then $x^*(x) = y^*(x)$. And so $y^{**} = \kappa(x)$.

So now we must verify (N). That is, $\kappa^{-1}(r^*(Y^{**})) \subset Y$.

Assume not. Then there exists $x \in \kappa^{-1}(r^*(Y^{**})) \setminus Y$. Now using that $Y \subset X$ is closed, by one of the corollaries of Hahn-Banach, there exists $x^* \in X^*$ such that $x^*(x) = 1$ and $x^*|_Y = 0$. Thus $r(x^*) = 0$. To get teh contradication, we write $x = \kappa^{-1}(r^*(y^{**}))$ for some $y^{**} \in Y^{**}$. Then $0 = y^{**}(r(x^*)) = r^*(y^{**})(x^*) = x^*(x) = 1$ contradiction. □

So now we finally prove the main theorem.

Only if: Assume that $A \subset X$ is a weakly sequentially compact set and that X is reflexive. Then we need to show that A is bounded. If not, then there exists a sequence contained in A with $|x_n| = n$ for all $n \geq 1$. By weak sequential compactness, we have $\{x_{n_k}\}$ a subsequence that converges weakly. By lemma 1 we have that $\{x_{n_k}\}$ is bounded, which is a contradiction.

If(hard): Use Cantor diagonal trick. Assume that X is reflexive and $A \subset X$ bounded. Let $\{y_n\} \subset A$ and let Y be the closure of the span of the y_i. By lemma 4, $Y = Y^{**}$. Since $(Y^*)^* \simeq Y$, and Y is separable, by lemma 2 we have that Y^* is separable. (This is where reflexivity is used).

Hence, choose a dense countable set $\{y_n^*\} \subset Y^*$. By the boundedness of $\{y_1^*(y_n)\} \subset \mathbb{C}$, we can choose a subsequence $\{y_{n,i}\}$, $\{n_{1i}\} \subset \mathbb{N}$ such that $y_1^*(y_{n_{1i}})$ converges. Then choose a subsequence $\{y_{2i}\}$ such that $y_2^*(y_{n_{2i}})$ converges. INductively, choose $\{y_{n_{ki}}\} \subset \{y_{n_{k-1,i}}\}$ such that $y_\ell^*(y_{n_{ki}})$ converges for $\ell \in \{1, 2, \ldots, \ell\}$.

Then the sequence $\{y_{n_{kk}}\}$ has teh property that $\{y_\ell^*(y_{n_{kk}}) \subset \mathbb{C}$ converges for all ℓ.

We can interpret this as saying that $\{\kappa(y_{n_{kk}})(y_\ell^*)\}$ converges for all $y_\ell 6*$. By lemma 3, we conclude that $\{\kappa(y_{n_{kk}})(y^*)\} \subset \mathbb{C}$ converges for all $y^* \in Y^*$ and the limit is defined as an element of $Y^{**} \simeq Y$. So $\lim_{k\to\infty} \kappa(y_{n_{kk}}) = y^{**} = k(y_0) \Rightarrow y_(y_{n_{kk}}) \to y^*(y_0)$ for all $y^* \in Y^*$. Thus, $\{y_{n_{kk}}\}$ weakly converges to y_0.

Corollary 12.6. *A reflexive space is weakly complete.*

Proof. If $\{x_n\}$ is a weakly cauchy sequence, then it is bounded. Now apply the main theorem. □

Examples of reflexive Banach Spaces:
Hilbert Spaces, $L^p(X)$ for $1 < p < \infty$.
Remarks on Hilbert Spaces in the Abstract

Lemma 12.7 (Cauchy-Schwarz Inequality)**.** $|(x,y)| \leq \sqrt{(x,x)}\sqrt{(y,y)}$.

Proof. $0 \leq (x + \alpha y, x + \alpha y) = (x,x) + |\alpha|^2(y,y) + \alpha(y,x) + \bar{\alpha}(x,y)$. This is $\|x\|^2 + |\alpha|^2\|y\|^2 + 2\Re(\alpha(y,x))$ Now set $\alpha = -(x,y)/\|y\|^2$. □

Lemma 12.8. *Let $x \in H$ and $K \subset H$ have the property that $\frac{1}{2}(K+K) \subset K$ That is, K is convex. Choose k_i in K such that $\lim_{i\to\infty} |x-k_i| = \inf_{k\in K} |x-k|$. Then $\{k_i\}$ converges.*

Proof. $|k_i - k_j|^2 = 2|x-k_i|^2 + 2|x-k_j|^2 - 4|x-(k_i+k_j)/2|^2 \stackrel{i,j\to\infty}{\to}$ stuff. □

Lemma 12.9. *The orthogonal complement $M^\perp$ of a closed linear subspace $M \subset H$ is a closed linear subspace, and $H = M \oplus M^\perp$.*

Proof. Closedness of $M^\perp$ follwos from Cauchy-Schwartz inequality

Now we must show that $M \oplus M^\perp$ is H. Assume that we are given $x \in H$. We need to show that there exists $m \in M$, $\tilde{m} \in M^\perp$ such that $x = m + \tilde{m}$. Choose an m such that $|x-m| = \inf |x-k|$ for $k \in M$. This is possibly by the lemma.

Define $\tilde{m} = x - m$. Need to show that $\tilde{m} \perp M$. To show that $\tilde{m} \in M^\perp$, take $|x-m-\alpha m_1| \geq \inf_{k\in M} |x-k| = |x-m|$ for some $\alpha \in \mathbb{C}$, $m_1 \in M$ arbitrary. Then $0 \leq |x-m-\alpha m_1|^2 - |x-m|^2 = |\alpha|^2|m_1|^2 - \alpha(m_1, x-m) - \bar{\alpha}(x-m, m_1)$ and set $\alpha = \lambda(x-m, m_1)$ for $\lambda \in \mathbb{R}$. Then $\lambda^2|m_1|^2|(x-m,m_1)|^2 - 2\lambda|(x-m,m_1)|^2 = 0$.

If $(x-m, m_1) \neq 0$, for some $m_1 \in M$, then choose λ small enough. Contradiction. □

13 Lecture 13

We are looking at Hilbert Spaces

Last time, we proved the lemma

Lemma 13.1. *$M \subset H$ a subspace which is closed. Then $H = M \oplus M^\perp$.*

Theorem 13.2 (Riesz Representation Theorem)**.** *For every $y^* \in H^*$, there exists a unique $y \in H$ such that $y^*(x) = \langle x, y\rangle$ for all $x \in H$.*

The map G taking $y^* \to y$ is a conjugate linear isometric isomorphism $H^* \to H$ with $|y^*| = |y|$.

Proof. Introduce $M = \{x \in H | y^*(x) = 0\}$. Then M is closed by the continuity of y^*, and if $y^* \neq 0$, it is a proper subspace. Therefore, there exists $y_1 \in M^\perp$.

$y^*(x - y^*(x)y_1/y^*(y_1)) = 0$ for all x, and so $\langle x - \frac{y^*(x)y_1}{y^*(y_1)}, y_1\rangle = 0$, and so $\langle x, y_1\rangle = y^*(x)\frac{\langle y_1, y_1\rangle}{y^*(y_1)}$.

Replace y_1 by $y = \alpha y_1$, with $\alpha = \frac{y^*\bar{(}y_1)}{\langle y_1, y_1\rangle}$ gives us $\langle x, y\rangle = y^*(x)$, so such a y exists.

For uniquenessm if also $y^*(x) = \langle x, \tilde{y}\rangle$, then $\langle x, y - \tilde{y}\rangle = 0$ for all $x \in H$, so $y = \tilde{y}$.

Now the isometric property: By Cauchy-Schwartz, we have $|\langle x, y\rangle| \leq |y||x|$, and $|y^*| \leq |Y|$, but also $\langle y, y\rangle = |y||y| \Rightarrow |y^*| \geq |y|$. □

Theorem 13.3. *A Hilbert Space H is reflexive. That is, the canonical embedding $\kappa : H \to H^{**}$ is onto.*

Proof. First equip H^* iwth teh following Hilbert Space structure $\langle x^*, y^*\rangle = \langle G(y^*), G(x^*)\rangle$, and now apply Riesz to H^*. Thus, given $y^{**} \in H^{**}$, $y^{**}(x^*) = \langle x^*, G(y^{**})\rangle$, can call $G(y^{**}) = y^*$. Then $y^{**}(x^*) = (x^*, y^*) = (G(y^*), G(x^*)) = x^*(G(y^*))$ and so $\kappa(G(y^*)) = y^{**}$. □

Example: $H = L^2(X, d\mu)$ has $H^* = L^2(X, d\mu)$.

We want to generalize this. $L^p(X)$ for $1 < p < \infty$ has $(L^p)^{\simeq} L^q$ where $1/p + 1/q = 1$, and so L^p is reflexive.

13.1 Radon-Nikodým Theorem

The absolute continuity of measures.

Definition 13.1 (Signed Measure)**.** *A signed measure ν on a σ-algebra M of subsets of X is a function $v : M \to \mathbb{R} \cup \{\infty\}$ satisfying*

1. *$v(E) \in (-\infty, \infty]$ for all $E \in M$*
2. *Countably additive.*

Example 13.1. *Let f be integrable in the extended sense and $f^- = \min\{f, 0\} \in L^1$. Then $v(E) = \int_E f d\mu$ is a signed measure.*

Definition 13.2 (Total Variation)**.** *Let ν be a signed measure, then the total variation $|v|$ is given by $|v|(E) = \sup_{\cup_{j=1}^\infty E_j = E, E_j \text{ disjoint}} \sum_{j=1}^\infty |v(E_j)|$*

Theorem 13.4. *Total variation is a positive measure satisfying $|v(E)| < |v|(E)$ for all E in M.*

Proof. Only countable additivity needs a proof.

Let $\{E_j\}_{j\geq 1} \subset M$ disjoint. We check that $\sum_{j\geq 1} |v|(E_j) \leq |v|(E)$, with $E = \cup E_j$. We also will check the converse inequality.

Choose for each j a number $\alpha_j < |v|(E_j)$. Hence, by definition, there exists a disjoint decomposition $E_j = \cup_{i\geq j} F_{ij}$ with $\alpha_j \leq \sum_i |v(F_{ij})|$, and hence $\sum \alpha_j \leq \sum_{ij} |v(F_{ij})|$, but $E = \cup_{ij} F_{ij}$, and so $\sum \alpha_j \leq |v|(E)$. Now let $\alpha_j \to |v|(E_j)$ for all j, and we get $\sum_{j\geq 1} |v|(E_j) \leq |v|(E)$.

Write $E = \cup F_k$, then $E_j = \cup_k (E_j \cap F_k)$, adn $F_k = \cup_j (F_k \cap E_k)$. So $\sum_k |v(F_k)| = \sum_k |\sum_j v(F_k \cap E_k)| \leq \sum_{jk} |v(F_k \cap E_J)| \leq \sum_j |v|(E_j)$ by switching order of summation, and so we are done. □

Given an arbitrary signed measure v, we can write $v = v^+ - v^-$ where $v^+ = \frac{1}{2}(|v| + v)$ and $v^- = \frac{1}{2}(|v| - v)$ with $v^\pm \geq 0$.

Thus, statements about signed measures can be reduced to statements about positive measures.

Definition 13.3 (σ-finite)**.** *A signed measure is called σ-finite if $|v|$ is. This means, $X = \cup_{j=1}^\infty E_j$, $|v|(E_j) < \infty$.*

If we take v, μ with μ positive and v signed measure, we want to decompose v into 'atomic part' singular with respect to μ and an absolutely continuous part with $v_a(E) = 0$ if $\mu(E) = 0$. $v(E) = \int_E f d\mu + \delta_0(E)$. If $X = \mathbb{R}$ and $d\mu$ is lebesgue measure, then f will be an L^1 function.

Definition 13.4 (Mutually Singular)**.** *Two signed measures ν, μ are called mutually singular if there exist disjoint sets $A, B \in M$ with $v(E) = v(E \cap A)$ and $\mu(E) = \mu(E \cap B)$. We write this as $\nu \perp \mu$*

Definition 13.5 (Absolutely continuous)**.** *ν a signed measure, μ positive, then ν is absolutely continuous with respect to μ proved that $\nu(E) = 0$ whenever $\mu(E) = 0$. This is written $\nu << \mu$.*

Remark 13.1. *If $\nu \perp \mu$ and $\nu << \mu$, then $\nu \equiv 0$.*

Example 13.2. *ν given by $\nu(E) = \int_E f d\mu$ for $f^- \in L^1(d\mu)$ is absolutely continuous with respect to μ.*

Lemma 13.5. *Let $|\nu|$ finite and $\nu << \mu$. Then $\forall \epsilon > 0$, $\exists \delta > 0$ such that $(*)$ $\mu(E) < \delta \Rightarrow |\nu|(E) < \epsilon$.*

Proof. Assume that (*) fails. Then $\exists \epsilon > 0$ such that $\forall n \geq 1$, there exists E_n with $\mu(E_n) < 2^{-n}$ but $|v|(E_n) \geq \epsilon$. Then consider $E^* = \limsup_{n\to\infty} E_n = \cap_{n=1}^\infty \cup_{k\geq n} E_k$.

$\mu(\cup_{k\geq n} E_k) \leq \sum_{k\geq n} \mu(E_k) \leq 2^{-n+1}$, and so $\mu(E^*) = \lim_{n\to\infty} \mu(\cup_{k\geq n} E_k) = 0$, and so $|v|(\cup_{k\geq n} E_k) \geq \epsilon$.

$|v|(E^*) = \lim_{n\to\infty} |v|(\cup_{k\geq n} E_k) \geq |\epsilon|$, and we need $|\nu|(X) < \infty$ for this. □

Theorem 13.6 (Radon-Nikodým)**.** *Let μ be a σ-finite positive measure on M and ν a σ-finite signed measure, then there exist unique signed measures v_a, v_s on M such that $v_a << \mu$, $v_s \perp \mu$ and $v = v_a + v_s$. Furthermore, we have $v_a(E) = \int_E f d\mu$ for some extended μ-integrable function f.*

Proof. Stepwise: first we assume that both μ, ν are finite and positive. The stategy, due to von Neumann, is to use Riesz Representation Theorem for Hilbert spaces.

Define $\rho = v + \mu$, and now introduce a functional $L^2(X, d\rho)$ given by $\ell(\phi) = \int_X \phi(x) d\rho(x)$ for $\phi \in: L^2(X, d\rho)$. We calim that ℓ is a continuous functional.

$|\ell(\phi)| \le (\int |\phi|^2 d\rho)^{1/2} (\int d\rho)^{1/2} = \|\phi\|_{L^2}(\rho(X))^{1/2}$. Hence, by Riesz, there is a $g \in L^2(d\rho)$ such that $\ell(\phi) = \int_X \phi(x) g(x) d\rho(x)$.

Claim: $0 \le g \le 1$ almost everywhere. To check this, use smart test functions for ϕ. That is, let $E \in M$, $\phi = \chi_E$. Then $\ell(\chi_E) = \int_X \chi_E d\nu = \nu(E) = \int_E g(x) d\rho$, so $\nu(E)/\rho(E) = \frac{\int_E g(x) d\rho}{\rho(E)}$, and if $|\{x | g(x) > 1\}| \ne 0$, then $E = \{x | g(x) > 1\}$ so we have this ratio is greater than 1, contradiction.

Hnece, by Riesz, there exists $g \in L^2(X, d\rho)$ with $\int_X \phi(x) d\nu(x) = \int_X \phi(x) g(x) d\rho(x) = \int_X \phi(x) g(x)(d\nu + d\mu)(x)$, $\int_X (\phi(x)(1 - g(x)) d\nu(x) = \int_X \phi(x) g(x) d\mu(x)$, and $A = \{x | g(x) < 1\}$, $B = \{x | g(x) = 1\}$. Then we define $\nu_a(E) = \nu(E \cap A)$ and $\nu_s(E) = \nu(E \cap B)$.

Claim: v_a is absolutely continuous.

$\phi(x) = \chi_{E \cap A}(1 + g + g^2 + \ldots + g^n)(x)$, and so $\int_X \phi(x)(1 - g(x)) d\nu(x) = \int_{E \cap A}(1 - g^{n+1}(x)) d\nu(x)$, and taking $n \to \infty$, we get $\int_{E \cap A} d\nu(x) = v_a(E)$.

$\int_X \phi(x) g(x) d\mu(x) = \int_{E \cap A} g(1 + g + \ldots + g^n) d\mu$ as $n \to \infty$, by monotonicity, this is $\int_{E \cap A} \frac{g}{1-g} d\mu$, and in particular, $\frac{g}{1-g} \chi_A \in L^1(d\mu)$. So $v_a(E) = \int_E \frac{g}{1-g} \chi_A d\mu \Rightarrow v_a(E)$ absolutely continuous, and so we have the desired representation. ☐

14 Lecture 14

Radon-Nikodým Theorem:

Step 1: Assume that ν, μ positive finite and set $\rho = \nu + \mu > \mu$. Then $0 \le g \le 1$, $\int_X f(x) d\nu(x) = \int_X g(x) d\mu(x)$ for all $f \in L^2(d\rho)$. iff $\int_X f(x)(1 - g(x)) d\nu(x) = \int_X f(x) g(x) d\mu(x)$.

$A = \{x \in X | g < 1\}$ and $B = \{x \in X | g = 1\}$, then $\mu(B) = 0$, and $v_a(E) = v(E \cap A)$ and $v_s(E) = v(E \cap B)$

Now $\nu_a(E) = \int_E \frac{g}{1-g} d\mu$ where $g/(1-g) \in L^1(d\mu)$.

Step 2: Assume that ν, μ are positive, σ-finite. $X = \cup E_i$ such that $\nu(E_i) + \mu(E_i) < \infty$. Then using step 1, we write $\nu_i(E) = \nu(E \cap E_i) = \nu_{i,a}(E) + \nu_{i,s}(E)$. Define $\nu_a = \sum \nu_{i,a}$ and $\nu_s = \sum \nu_{i,s}$, we have $\nu = \nu_a + \nu_s$.

$\nu_a(E) = \int_E \sum f_i(x) d\mu(x)$ where each f_i is produced for $\nu_{i,a}$ by step 1. Take $f = \sum f_i$, integrable in the extended sense.

Step 3: If ν is signed, then $\nu = \nu^+ - \nu^-$.

Theorem 14.1 (Vitali's Theorem). *Let (X, M, μ) be a measure space, μ positive and if $A, B \in M$, then set $d(A, B) = \arctan(A \setminus B \cup B \setminus A)$. Then $M/\sim$ with $A \sim B \iff d(A, B) = 0$ is a metric space. (Exercise)*

Lemma 14.2. *$M/\sim$ is a complete metric space.*

Proof. $\{E_n\}_{n\geq 1}$ is a Cauchy sequence, then choose a subsequence $\{E_{n_i}\}$ such that $d(E_n, E_{n_i}) < 2^{-i}$ for $n \geq n_i$.

Exercise: Show that $E_n \to E_*$, where $E_* = \cup_i \cap_{j\geq i} E_{n_j}$. □

Definition 14.1 (μ-continuous)**.** *A scalar valued set function $\lambda : M \to \mathbb{C}$ is called μ-continuous if $\forall \epsilon > 0$, there exists $\delta > 0$ such that $|\lambda(E)| < \epsilon$ if $\mu(E) < \delta$.*

Observation: Such a function descends to a function $\lambda : M/\sim \to \mathbb{C}$ proved that λ is additive.

Indeed, it becomes a continuous function on $M/\sim$. To see it, $E_m \to E$ in $M/\sim$ implies that $\mu(E_n \backslash E) \to 0$ and $\mu(E \backslash E_m) \to 0$ and so $\lambda(E \backslash (E_n \cap E_m)) \to 0$ and $\lambda(E \setminus (E_n \cap E_m)) \to 0$

And so $\lambda(E) - \lambda(E_m) = \lambda(E \setminus E_m \cap E) - \lambda(E_m \setminus E_m \cap E) \to 0$.

Lemma 14.3. *The set operation $A, B \to A \cup B, A \cap B, A \Delta B$ are well-defined and continuous on $(M/\sim, d)$*

Proof. Exercise □

Theorem 14.4. *Let (X, M, μ) be a measure space and $\{\lambda_n\}$ be μ-continuous additive set functions on M. If $\lim_{n\to\infty} \lambda_n(E)$ exists for all $E \in M$, then $\lim_{\mu(E)\to 0} \lambda_n(E) = 0$ uniformly in n.*

That is, for all $\epsilon > 0$, there exists $\delta > 0$ such that $|\lambda_n(E)| < \epsilon$ if $\mu(E) < \delta$ for all n

Proof. Application of Baire.

λ_n discends to a continuous function on $(M/\sim, d)$ for all n, and hence for all $\epsilon \geq 0$, the sets $A_{n,m} = \{E \in M|, |\lambda_n(E) - \lambda_m(E)| \leq \epsilon\}$ are closed, as well as $A_p = \cap_{n,m\geq p} A_{n,m}$

Since $\lim_{n\to\infty} \lambda_n(E)$ exists, for all $E \in M$, we have $M/\sim = \cup_{p\geq 1} A_p$. By Baire, at least one of the A_p has nonempty interior. So there exists $q \in \mathbb{N}$ and $r > 0$ such that $|\lambda_n(E) - \lambda_m(E)| \leq \epsilon$ provided $\mu(E \Delta A) < r$ for some $A \in M$, $n, m \geq q$.

Now, choose δ with $0 < \delta < r$ such that $|\lambda_n(B)| < \epsilon$ whenever $\mu(B) < \delta$ for $n = 1, 2, \ldots, q$.

Claim: If $B \in M$, then $\mu(B) < \delta$ implies that $\lambda_n(B) < 3\epsilon$ for all n.

For $n \geq q$:

$$\begin{aligned}
|\lambda_n(B)| &= |\lambda_q(B) + \lambda_n(B) - \lambda_q(B)| \\
&= |\lambda_q(B) + \lambda_n(A \cup B) - \lambda_q(A \cup B) - [\lambda_n(A \setminus B) - \lambda_q(A \setminus B)] \\
&\leq \lambda_q(B)| + |lambda_n(A \cup B) - \lambda_q(A \cup B)| + |\lambda_n(A \setminus B) - \lambda_q(A \setminus B)| \\
&< 3\epsilon
\end{aligned}$$

□

Theorem 14.5 (Vitali)**.** *Let $1 \leq p \leq \infty$, (X, M, μ) a measure space and $\{f_n\} \subset L^p(d\mu)$ such that $f_n \to f$ pointwise almost everywhere. Then f is in L^p and $|f_n - f|_{L^p} \to 0$ if and only if*

1. $\lim_{\mu(E)\to 0}\int_E |f_n(x)|^p d\mu = 0$ *uniformly in* n

2. $\forall \epsilon > 0$, *there exists* $E_\epsilon \in M$ *such that* $\mu(E_\epsilon) < \infty$, $\int_{E_\epsilon^c} |f_n|^p d\mu < \epsilon$ *for all* n.

Proof. If: First show that for all $\epsilon > 0$, there exists A_ϵ such that $f_n|_{A_\epsilon} \to f|_{A_\epsilon}$ uniformly and $\|f_n|_{A_\epsilon^c}\|_{L^p} < \epsilon$ and $\mu(A_\epsilon) < \infty$.

Proof with hint will be exercise. $\square$

Theorem 14.6 (Structure of the dual of $L^p(d\mu)$)**.** . *Let* $1 \le p \le \infty$, $1/p+1/q = 1$, *then there is an isometric isomorphism between* $(L^p(d\mu))^*$ *and* $L^q(d\mu)$ *via the relation* $x^*(f) = \int_X g(x)g(x)d\mu$ *for all* $f \in L^p(d\mu)$ *with* $g(x) \in L^q(d\mu)$.

Proof. Fairly easy to check that each $g \in L^q(d\mu)$ defines an $x^* \in (L^p(d\mu))^*$ via this map, because continuity of x^* is equivalent to boundedness of x^*, and by Hölder's Inequality, $|\int g(x)f(x)d\mu| \le \|g\|_{L^q}\|f\|_{L^p}$. Challenge: Show that each x^* is given by a suitable $g \in L^q(d\mu)$.

Step 1: Assume that $\mu(X) < \infty$. Given $x^* \in (L^p(d\mu))^*$, we'll introduce a signed measure, apply Radon-Nikodým.

$E \in M$ maps to $x^*(\chi_E)$. If $E = \cup E_j$, with E_j disjoint, then $\chi_E = \sum \chi_{E_i}$, in L^p.

$x^*(\chi_E) = x^*(\chi_{E_i}) = \lim_{N\to\infty} x^*(\sum_{i=1}^N \chi_{E_i}) = \lim_{N\to\infty}\sum_{i=1}^N x^*(\chi_{E_i}) = \sum x^*(\chi_{E_i})$.

This implies that $E \to x^*(\chi_E)$ is a signed measure. Further, since $\|\chi_E\|_{L^p} \to 0$ as $\mu(E) \to 0$, by Radon-Nikodým, there exists a $g \in L^1$ such that $x^*(\chi_E) = \int_X g(X)\chi_E(x)d\mu$.

Then it is clear that also $x^*(f) = \int_) Xg(x)f(x)d\mu$ for all f a μ-simple function. Given a general $f \in L^p(d\mu)$, there exists $f_n \in L^p(d\mu)$, μ-simple functions such that $f_n \to f$ almost everywhere and in $L^p(d\mu)$. Then also $gf_n \to gf$ almost everywhere.

We want to use Vitali to conclude that $gf_n \to gf$ in the L^1 sense. By finiteness of $\mu(X)$, the 2nd condition of Vitali is trivially satisfied. We need to check that $\lim_{\mu(E)\to 0}\int_E g(x)f_n(x)d\mu = 0$ uniformly in n.

To see this, introduce the set functions $\lambda_n(E) = \int_E gf_n d\mu$. Note that $\lim \lambda_n(E)$ exists for all $E \in M$.

So $gf_n \to gf$ in L^1 norm. As $gf \in L^1$ and $\lim_{n\to\infty}\int_X gf_n d\mu = \int_X gf d\mu$, we have $\lim_{n\to\infty} x^*(f_n) = x^*(f)$.

All that remains is to conclude that $g \in L^q$. We use the "bootstrapping" procedure:

The function taking $z \to e^{i\theta}$ for $z \neq 0$ (that is, $z/|z|$) and taking $0 \to 0$. Take $\text{Arg}(z)$ to be theta and $g_1 = |g(-)|^{1/p}\,\text{Arg}(g(-))$. This is in L^p.

So $x^*(g_1) = \int_X |g|^{1+1/p}d\mu \le |x^*||g_1|_{L^p} = |x^*|\left(\int_X |g|d\mu\right)^{1/p} = |x^*|(x^*(\text{Arg}\,g))^{1/p} \le |x^*|^{1+1/p}(\mu(X))^{1/p}$ and so $g \in L^{1+1/p} \to L^{1+1/p+1/p^2} \to \dots$ $\square$

15 Lecture 15

$(L^p(d\mu))^*$

Our tools are as follows:

Theorem 15.1 (Vitali-Hahn-Saks)**.** *Let (X, M, μ) be a measure space and $\{\lambda_n\}$ a set of μ-continuous additive set functions. If $\lim_{n\to\infty} \lambda_n(E)$ exists for all $E \in M$, then $\lim_{\mu(E)\to 0} \lambda_n(E) = 0$ uniformly in n if and only if $\forall\epsilon > 0$, there exists $\delta > 0$, such that if $\mu(E) < \delta$ then $|\lambda_n(E)| < \epsilon$ for all $n \geq 1$.*

Theorem 15.2 (Vitali)**.** *Let $1 \leq p < \infty$ and (X, M, μ) a measure space with $\{f_n\} \subset L^p(d\mu)$ and $f_n \to f$ pointwise. Then $f \in L^p$ and $f_n \to f$ in L^p sense iff*

1. *$\lim_{\mu(E)\to 0} \int_E |f_n|^p d\mu = 0$ uniformly in n*

2. *$\forall\epsilon > 0$, $\exists E_\epsilon \in M$ with $\mu(E_\epsilon) < \infty$ such that $\int_{E_\epsilon^c} |f_n|^p d\mu < \epsilon$ for all n.*

Theorem 15.3 (Main Theorem: Structure of $(L^p(d\mu))^*$)**.** *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then there is an isometric isomorphism between $(L^p(d\mu))^*$ and $L^q(d\mu)$ via the relation $x^*(f) = \int_X g(x)f(x)d\mu$ for all $f \in L^p(d\mu)$ and some $g \in L^q(d\mu)$ and $|x^*| = \|g\|_{L^q}$.*

Proof. Step 1: Assume that $\mu(X) < \infty$. Let $x^* \in (L^p(d\mu))^*$. We define a signed measure via $\nu(E) = x^*(\chi_E)$. This is well defined, because $\|\chi_E\|_{L^p} \leq \|1_X\|_{L^p} = (\mu(X))^{1/p}$. To check that $\nu(E)$ is a measure, assume that $E = \coprod E_j$, then $\sum \chi_{E_i} = \chi_E$ in L^p sense, and so it works out.

We claim that ν is μ-continuous or $\nu << \mu$. $\nu(E) \leq (\mu(E))^{1/p}|x^*| \Rightarrow \nu(E) = 0$ if $\mu(E) = 0$, and Radon-Nikodým says $x^*(\chi_E) = \nu(E) = \int_X g(x)\chi_E(x)d\mu$ for some $g \in L^1(d\mu)$. So $x^*(f) = \int_X g(x)f(x)d\mu$ for any μ-simple function.

Now let's assume that $f \in L^p(d\mu)$. Then choose a sequence $\{f_n\}$ of μ-simple functions with $f_n \to f$ almost everywhere and with respect to L^p. By continuity of x^*, $x^*(f_n) \to x^*(f)$. And $x^*(f_n) = \int_X g(x)f_n(x)d\mu$. So $g(x)f_n(x) \to g(x)f(x)$ a.e. and want to show that this convergence is also in the L^1-sense.

Here we use the technical theorem that $\int_E g(x)f_n(x)d\mu = x^*(\chi_E f_n) \to x^*(\chi_E f)$ for all $E \in M$.

Now we introduce a family of set functions $\nu_n(E) = \int_E g(x)f_n(x)d\mu$, and $\lim_{n\to\infty} \nu_n(E)$ exists for all $E \in M$ and so by Vitali-Hahn-Saks, $\lim_{\mu(E)\to 0} \int \nu_n(E)d\mu = 0$.

And so, we apply Vitali's Tehorem to $\{g(x)f_n(x)\}$ and so we have $g(x)f_n(x) \to g(x)f(x)$ in L^1. So $gf \in L^1(d\mu)$ and $x^*(f) = \int g(x)f(x)d\mu$. Make judicious choice for f and define $g_1(x) = |g(x)|^{1/p} \operatorname{Arg}(g(x))$ which is in L^p. Now $x^*(g_1) = \int_X |g|^{1+1/p}d\mu \leq |x^*|\|g\|_{L^p} = |x^*|\left(\int |g(x)|d\mu\right)^{1/p}$, which is less than or equal to $|x^*|(|x|^*\mu(X))^{1/p} = |x^*|^{1+1/p}(\mu(X))^{1/p}$.

Thus, $g \in L^{1+1/p}$. Now we define $g_2 = |g|^{(1+1/p)/p} \operatorname{Arg}(g)$.

Now $\|g_2\|_{L^p} = \left(\int_X |g(x)|^{1+1/p}d\mu\right)^{1/p} \leq |x^*|^{1/p+1/p^2}(\mu(X))^{1/p^2}$, and so $\int_X |g(x)|^{1+1/p+1/p^2}d\mu = \int g(x)|g|^{1/p+1/p^2} \operatorname{Arg}(g(x))d\mu = x^*(g_2) \leq |x^*|\|g_2\|_{L^p} = |x^*|^{1+1/p+1/p^2}(\mu(X))^{1/p}$.

Now proceed inductively to define $g_n = |g|^{1/p+\ldots+1/p^n}\,\mathrm{Arg}(g(x)).\int_X |g|^{1+\ldots+1/p^n} d\mu \leq |x^*|^{1+\ldots+1/p^n}(\mu(X))^{1/p^n}$.

Note that $\sum_{i=0}^{\infty}\frac{1}{p^i} = q$, and so by Fatou's Lemma, as $|g|^{1+\ldots+1/p^n} \to |g|^q$, we have $\int |g|^q d\mu \leq |x^*|^q \Rightarrow \|g\|_{L^q} \leq |x^*|$, and by Hölder's Inequality, we also have $|x^*| \leq \|g\|_{L^q}$, and so $|x^*| = \|g\|_{L^q}$.

Step 2: Now assume that $\mu(X) = \infty$. Define $M_1 = \{E \in M | \mu(E) < \infty\}$ (note, not a σ-algebra, but it is an algebra). For $E \in M_1$, denote $L^p(E) = \{f \in L^p(d\mu) | f|_{E^c} \equiv 0\}$.

Then $x^*_E = x^*|_{L^p(E)}$, and by what we've shown, for all $E \in M_1$, there exists $g_E \in L^q(E)$ such that $x^*_E(f) = \int_X g_E(x)f(x)d\mu$ for all $f \in L^p(d\mu)$. And so $\|g_E\|_{L^q} \leq |x^*|$. Further, if $E_1, E_2 \in M_1$, then $g_{E_1}|_{E_1\cap E_2} = g_{E_2}|_{E_1\cap E_2} \to \int_{E_1\cap E_2} g_{E_1} f d\mu = \int_{E_1\cap E_2} g_{E_2} f d\mu$.

Thus, $\nu(E) = \|g_E\|^q_{L^q(E)} \leq |x^*|^q$ increasing set function. Hence there exists an increasing sequence $E_n \in M_1$ such that $|x^*_{E_n}| \to \sup_{E\in M_1}|x^*_E| \leq |x^*|$.

Since $g_{E_n} = g_{E_{n+k}}$ for all $k \geq 0$, almost everywhere with respet to μ on E_n, and so $g(x) = \lim_{n\to\infty} g_n(x)$ defined almost everywhere on $\cup_{n=1}^{\infty} E_n$.

By the monotone convergence theorem, $\|g\|_{L^q} = \lim_{n\to\infty}\|g_{E_n}\|_{L^p} = \sup_{E\in M_1}|x^*_E| \leq |x^*|$.

Our claim now is that g is the desired representative, $\int_X fg d\mu = x^*(f) \forall f \in L^p(d\mu)$.

To see this, first assume $E \in M_1$, $E \cap F = \emptyset$ (where $F = \cup E_n$). $\|g_{E\cup E_n}\|^q_{L^q} = \|g_E\|^q_{L^q} + \|g_{E_n}\}^q_{L^q} \Rightarrow g_E = 0$ almost everywhere.

Hence, if $f \in L^p(E)$ and $E \in M_1$ arbitrary, then $x^*(f) = \chi^*_E(f) = \int_E g_E(x)f(x)f\mu = \int_{E\setminus F} g_E(x)f(x)d\mu + \int_{E\cap F} g_E(x)f(x)d\mu = \int_X g(x)f(x)d\mu$. We conclude by observing that $\cup_{E\in M_1} L^p(E)$ is dense in $L^p(d\mu)$. $\square$

Corollary 15.4. *For $1 < p < \infty$ the space $L^p(d\mu)$ is reflexive, and hence bounded subsets are weakly sequentially compact.*

Proof. Let $x^{**} \in (L^p)^{**} = (L^q)^*$. Hence there exists $y^* \in (L^q(d\mu))^*$ such that $x^* * (x^*) = y^*(g)$ if x^* is represented as $x^*(f) = \int_X g(x)f(x)d\mu$. But then there is $h \in L^p$ such that $y^*(g) = \int_X h(x)g(x)d\mu = x^*(h)$, and so $x^{**} = \kappa(h)$. $\square$

Remark: It is true, provided that (X, M, μ) is σ-finite, that $(L^1)^* = L^\infty$, however, $(L^\infty)^* \not\cong L^1$.

15.1 Distribution Theory

How to make sense of things like Δu without assuming that $u \in C^2$.

If $\phi \in C_0^\infty(\mathbb{R}^n)$, then $\langle \Delta u, \phi\rangle = \langle u, \Delta\phi\rangle$. So in a sense, distributions are in the dual of $C_0^\infty(\mathbb{R}^n)$ (but not really, because it's not a Banach space) There are so many distributions because C_0^∞ is a VERY restricted space.

Definition 15.1 (Distribution)**.** *Let $X \subset \mathbb{R}^n$ open. Then a distribution u is a linear functional on $C_0^\infty(X)$ with the property that for every $K \subset X$ compact with nonempty interior (written $K << \mathbb{R}^n$), there exist $C \in \mathbb{R}, k \in \mathbb{N}$ such that $|u(\phi)| \leq C\sum_{|a|\leq k}\sup|\partial^a\phi|$ for all $\phi \in C_0^\infty(K)$.*

Example 15.1. *Let $u \in L^2(\mathbb{R}^n)$. Define Δu by its action on $C_0^\infty(\mathbb{R}^n)$ via $\langle \Delta u, \phi\rangle = \langle u, \Delta\phi\rangle$. We claim that Δu is a distribution.*

Pick a $K << \mathbb{R}^n$ and let $\phi \in C_0^\infty(K)$. Then $|\langle \Delta u, \phi\rangle| \leq \|u\|_{L^2}\|\Delta\phi\|_{L^2} \leq \|u\|_{L^2} \sum \|\partial^a \phi\|_{L^\infty}(m(K))^{1/2}$. Then $k = 2$ and $C = \|u\|_{L^2}(m(K))^{1/2}$.

Example 15.2. *δ_{x_0} is defined by $\langle \delta_{x_0}, \phi\rangle = \phi(x_0)$, and derivatives of δ_{x_0} by $(-1)^|\alpha|\langle \delta_{x_0}, \partial^\alpha \phi\rangle$.*

Other examples: every $f \in L^p$ for some $1 \leq p \leq \infty$ is a distribution.

Definition 15.2 (Finite Order)**.** *A distribution is of finite order if the k from the definition of the distribution is independent of K.*

16 Lecture 16

References: $L^p(d\mu)$ stuff: Chapter in Dunford-Schwarz "Linear Operators Vol I"

Riesz Representation for Functions on $C^0(\mathbb{R}^n)$ in Rudin "Real and Complex Analysis"

Distribution Theory Hörmander "Partial Differentail Operators Vol I"

Last time, we introduced distributions.

The space of distributions is denoted $D'(X)$.

Theorem 16.1. *Let $u \in D'(X)$, $X \in \mathbb{R}^n$ open. Assume $u(\phi) \geq 0$ whenever $\phi \in C_0^\infty(X)$ is nonnegative. Then u is a positive measure μ on X such that $\mu(K) < \infty$ for all $K << X$ by $u(\phi) = \int_X \phi(x)d\mu(x)$ for all $\phi \in C_0^\infty(X)$. In particular, u is of order 0.*

Strategy, show that u extends to $C_0^0(X)$ and then use Riesz Representation Theorem.

We will need the following lemma:

Lemma 16.2 (Smooth Urysohn)**.** *Given $K \subset X$ compact, then there exists $\chi \in C_0^\infty(X)$ such that $\chi \equiv 1$ on X, $0 \leq \chi \leq 1$.*

Proof. Let $\inf_{x\in K, y\in X^c} |x-y| \geq 4\epsilon > 0$ for suitable $\epsilon > 0$ (note that $X \subset \mathbb{R}^n$). Then choose $\tilde{\chi} \in C_0^\infty(B_1(0))$, $\int \tilde{\chi}(x)dx = 1$, $0 \leq \tilde{\chi}$ and $\tilde{\chi}_\epsilon = \epsilon^{-n}\tilde{\chi}(x/\epsilon)$, then $\operatorname{supp} \tilde{\chi}_\epsilon \subset B_\epsilon(0)$, and $\int \tilde{\chi}_\epsilon(x)dx = 1$.

Define $\chi_{K_{2\epsilon}}$ to be the characteristic function of the set $K_{2\epsilon} = \{y \in \mathbb{R}^n | \inf_{x\in K} |x-y| \leq 2\epsilon\}$.

Set $\chi(x) = [\chi_{K_{2\epsilon}} * \chi_\epsilon](x) = \int_{\mathbb{R}^n} \chi_{K_{2\epsilon}}(x-y)\tilde{\chi}_\epsilon(y)dy$.

Then $\operatorname{supp}\chi \subset K_{3\epsilon}$ and $1 \geq \chi(x) \geq 0$, $\chi \in C_0^\infty(X)$. We also have $(1-\chi) = (1-\chi_{K_{2\epsilon}}) * \tilde{\chi}_\epsilon$.

If $x \in K$, then since $y \in B_\epsilon(0)$ on $\operatorname{supp}\tilde{\chi}_\epsilon$, so $x - y \in K_\epsilon$ on support of integrand, so $(1-\chi_{K_{2\epsilon}}))(x-y) = 0$ if $x \in U$, $y \in$ support of integral implies that $\chi_K \equiv 1$. $\square$

Now we prove the theorem.

Proof. Given $\phi \in C_0^\infty(X)$, let $K = \operatorname{supp}\phi << X$ and choose χ as inn the lemma. Then $\chi \sup|\phi| \pm \phi \geq 0$, if ϕ real valued by the positivity of u.

$u(\chi \operatorname{supp}|\phi| \pm \phi) \geq 0$ implies that $|u(\phi)| \leq u(\chi)\sup|\phi|$, and so $\forall$ compact $K \subset X$, there exists $C(K)$ such that for all $\phi \in C_0^\infty(K) \subset C_0^\infty(X)$ $|u(\phi)| \leq C(K)\sup|\phi|$.

Next assume that ϕ is complex valued. Then choose $\theta \in \mathbb{R}$ such that $e^{i\theta}u(\phi) \in \mathbb{R}$. Then $e^{i\theta}u(\phi) = u(e^{i\theta}\phi) = u(\Re(e^{i\theta}\phi)) + iu(\Im(e^{i\theta}\phi))$ and by preceeding, $|u(\Re(e^{i\theta}\phi))| \leq C(K)\sup|e^{i\theta}\phi| = C(K)\sup|\phi|$.

Thus, for all $K << X$, $\phi \in C_0^\infty(K)$, there exists $C(K)$ such that $|u(\phi)| \leq C(K)\sup|\phi|$.

By approximating arbitrary functions $\psi \in C_0^0(X)$ by $C_0^\infty(X)$ functions (with support slight, but fixed, englargement of suppose ψ) we have that u extends to a continuous linear function on $C_0^0(X)$. Now conclude via Riesz. □

Topology on $D'(X)$. Use the "weak topology" which has as a basis for open sets $U_{\phi_1,\ldots,\phi_n,\epsilon,v} = \{u \in D'(X) : |u(\phi_i) - v(\phi_i)| < \epsilon \text{ for all } i \in \{1,\ldots,n\}\}$ where v is a fixed element in $D'(X)$, $\phi_1,\ldots,\phi_n \in C_0^\infty(X)$.

In the weak topology, a set A of distributions is open iff it can be written as a union of such sets.

Equivalently, a sequence of distributions $u_i \to u$ iff $u_i(\phi) \to u(\phi)$ for all $\phi \in C_0^\infty(X)$.

Theorem 16.3 (Weak Completeness Property for $D'(X)$)**.** *If $\{u_j\}_{j\geq 1} \subset D'(X)$ satisfies $u(\phi) = \lim_{j\to\infty} u_j(\phi)$ exists for all $\phi \in C_0^\infty(X)$, then $u \in D'(X)$ and $u_j \to u$ in the sense of distributions.*

Proof. We need to check that u satisfies the boundedness requirement for a distribution. Pick a compact set $K << X$. We shall equip $C_0^\infty(K)$ with the structure of a Frechet Space. Introduct the semi-norms $\|\phi\|_\alpha = \sup_K |\partial^\alpha \phi|$ for $\alpha = (\alpha_1,\ldots,\alpha_n)$ a multi-index.

Define $|\phi| = \sum_\alpha (2n)^{-|\alpha|}\frac{\|\phi\|_\alpha}{1+\|\phi\|_\alpha}$. Now (exercise) check that $(C_0^\infty(K), |\cdot|)$ is complete.

Further, each u_j is continuous with respect to the Frechet Space structure.

By the principle of uniform boundedness, for all $\epsilon > 0$, there exists $\delta > 0$ such that $|\phi| < \delta \Rightarrow |u_j(\phi)| < \epsilon$ for all $j \geq 1$.

In particular, also $|u(\phi)| \leq \epsilon$, and so u is a distribution on $C_0^\infty(K)$. To see this, assume not. Then for all $k \geq 1$, there exists $\phi_k \in C_0^\infty(K)$ such that $|u(\phi_k)| \geq 2^k \sum |\partial^\alpha \phi_k|$ by normalizing, we can require that $|u(\phi_k)| = 1$ and so $\sum |\partial^\alpha \phi_k| \leq 2^{-k}$, and this implies that $|\phi| < \delta$ for k large enough. Thus $|u(\phi_k)| < \epsilon$, contradiction. □

Example 16.1. *Limit of distributions:* $u_t(x) = \begin{cases} te^{itx} & x > 0 \\ 0 & x \leq 0 \end{cases}$ *and* $\langle u_t, \phi\rangle = \int_{\mathbb{R}} u_t(x)\phi(x)dx$ *for* $\phi \in C_0^\infty(\mathbb{R})$.

Question: What is $\lim_{t\to\infty} u_t(x) \in D'(\mathbb{R})$*?*

$u_t(\phi) = \int_0^\infty te^{itx}\phi(x)dx = i\phi(0) + i\int_0^\infty e^{itx}\phi'(x)dx = i\phi(0) - \phi'(0)/t - \int_0^\infty \frac{e^{itx}}{t}\phi''(x)dx \to i\phi(0)$ as $t \to \infty$.
And so $\lim_{t\to\infty} te^{itx} = i\delta_0$.

Important Operations on Distributions:

1. Localization: $U \to D'(U)$
2. Differentiation: $U \subset V$, a map $D'(V) \to D'(U)$

Theorem 16.4. *If $u \in D'(X)$ and every point in X has a neighborhood on which the restriction of u vanishes, then $u = 0$*

Theorem 16.5. *Let $\{X_i\}_{i\in I}$ arbitrary family of open sets in $\mathbb{R}^n$ such that $X = \cup X_i$, if $u_i \in D'(X_i)$ and $u_i|_{X_i\cap X_j} = u_j|_{X_i\cap X_j}$ for all i, j, then there exists a unique $u \in D'(X)$ such that $u_i = u|_{X_i}$.*

Corollary 16.6. *The map $U \to D'(U)$ is a sheaf.*

We need a lemma

Lemma 16.7. *Let $X_1, \ldots, X_k \subset \mathbb{R}^n$ open, $\phi \in C_0^\infty(\cup X_i)$. Then there exists $\phi_j \in C_0^\infty(X_j)$ for each j such that $\phi = \sum \phi_j$.*

Proof. Choose compact sets $K_j \subset X_j$ with $\operatorname{supp}\phi \subset \cup K_i$. By our smooth Urysohn Lemma, find $\psi_j \in C_0^\infty(X_j)$ with $0 \leq \psi_j \leq 1$ and $\psi_j \equiv 1$ on K_j. Then consider $\phi_1 = \phi\psi_1$, $\phi_2 = \phi\psi_2(1-\psi_1)$ etcetera. Then $\sum \phi_j - \phi = -\phi\prod_{j=1}^k(1-\psi_j) = 0$. □

And now we prove the first theorem:

Proof. Given $\phi \in C_0^\infty(X)$, find for each $x \in \operatorname{supp}\phi$ a nieghborhood $U_x \subset X$ such that $u|_{U_x} \equiv 0$. Finitely many of the U_x cover $\operatorname{supp}_{\phi_n}$ and call them $U_1, \ldots, U_k$. Then finte $\phi_1, \ldots, \phi_k$ and suppose $\operatorname{supp}\phi_i \subset U_i$. Then $\sum \phi_i = \phi$.
$u(\phi) = \sum u(\phi_i) = 0$ □

17 Lecture 17

Now we will prove the second statement.

Proof. The uniqueness follows from the first statement.
Assume that $\phi = \sum \phi_i$ for $\phi_i \in C_0^\infty(X_i)$. Then we necessarily have $u(\phi) = \sum u_i(\phi_i)$. We need to show that this canonically defines u.
Equivalently, we need to verify that whenever $\sum_{i=1}^M \phi_i \equiv 0$ for $\phi_i \in C_0^\infty(X_i)$, then $\sum_{i=1}^M u_i(\phi_i) = 0$.
Put $K = \cup\operatorname{supp}(\phi_i) << X$. Choose finitely many functions $\psi_k \in C_0^\infty(X_k)$ such that $\sum \psi_k|_K \equiv 1$.
Then $\psi_k\phi_i \in C_0^\infty(X_k \cap X_i)$, and by assumption $u_i(\psi_k\phi_i) = u_k(\psi_k\phi_i)$.
$\sum_i u_i(\phi_i) = \sum_{i,k} u_i(\phi_i\psi_k) = \sum_{i,k} u_k(\phi_i\psi_k) = \sum_k u_k(\sum_i \psi_k\phi_i) = 0$. □

Still need to check that u satisfies the bounds required of a distribution. Choose $L << X$ compact and let $\psi_k \in C_0^\infty(X_k)$ be finitely many k such that $\sum \psi_k|_L \equiv 1$.

If $\phi \in C_0^\infty(L)$, then $u(\phi) = \sum_k u(\psi_k\phi) = \sum_k u_k(\psi_k\phi)$.

For all k, there exists C_L^k and $P_{L,k} \in \mathbb{N}$, such that $|u_k(\psi_k\phi)| \leq C_L^k \sum_{|\alpha| \leq P_{L,k}} \sup_K |\partial^\alpha(\psi_k\phi)|$, and so there exists $c \in \mathbb{R}$ such that $|u(u)| \leq c \sum \sup |\partial^\alpha \phi|$, and so $u \in D'(X)$.

There are two very important classes f distributions.

1. Compactly Supported Distributions
2. Tempered distributions

Definition 17.1 (Support)**.** *Let* $u \in D'(X)$*. Then* $\operatorname{supp} u = \{x \in X | \exists U_x \ni x$ *such that* $u|_{U_x} \equiv 0\}^c$.

Example 17.1. *If* $f \in L^1_{Loc}(X)$*, then* $\operatorname{supp} f$ *in the sense of a distribution is essetially* $\operatorname{supp} f$ *as a measureable function.*

Example 17.2. *If* $u = \delta_0$*, then* $\operatorname{supp} u = \{0\}$.

u is compactly supported if $\operatorname{supp} u << X$.

Assume that $u \in D'(\mathbb{R}^n)$ is compactly supported. Then u extends canonically to a homomorphism $C^\infty(\mathbb{R}^n) \to \mathbb{C}$. Indeed, given $\phi \in C^\infty(\mathbb{R}^n)$, choose $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\psi|_{\operatorname{supp} u} \equiv 1$. Then define $u(\phi) = u(\phi\psi)$. This is independent of the choice of ψ, since if $\tilde{\psi}$ is another, then $u(\phi(\psi - \tilde{\psi})) = 0$.

If $K = \operatorname{supp} u$, then $u : C^\infty(\mathbb{R}^n) \to \mathbb{C}$ satisfies $|u(\phi)| \leq C \sum_{|\alpha| \leq k} \operatorname{supp}_K |\partial^\alpha \phi|$.

Conversely, if $u : C^\infty(\mathbb{R}^n) \to \mathbb{C}$ satisfies a relation of this type, it restricts to a compactly supported distribution $D'(\mathbb{R}^n)$.

So, if $C^\infty(\mathbb{R}^n)$ is equipped with a suitable Frechet space structure, then the linear functionals that are continuous $C^\infty(\mathbb{R}^n) \to \mathbb{C}$ are exactly the compactly supported distribution.

To equip $C^\infty(\mathbb{R}^n)$ with Frechet space structure, introduce the seminorms $\phi \to p_{n,\alpha}(\phi) := \sup_{K_n} |\partial^\alpha \phi|$ where K_n is compact for all n and $\cup K_n = \mathbb{R}^n$.

Define $|\phi| = \sum_{n,\alpha} (2d)^{-n-|\alpha|} \frac{p_{n,\alpha}(\phi)}{1+p_{n,\alpha}(\phi)}$.

Observation: $u : (C^\infty(\mathbb{R}^n), |\cdot|) \to \mathbb{C}$ is continuous iff there exists $k \in \mathbb{N}, C \in \mathbb{R}$, $K << \mathbb{R}^n$ such that $|u(\phi)| \leq C \sum_{|\alpha| \leq k} \sup_K |\partial^\alpha \phi|$.

Proof. Only if: If not, then there exist ϕ_n for all n with $|u(\phi_n)| \geq 100^n \sum_{|\alpha| \leq n} \sup_{\cup_{i=1^n} K_n} |\partial^\alpha \phi|$.

Rescaling, we may assume that $|u(\phi_n)| = 1$ and so $|\phi_n| < 2^{-n}$ if n large enough, contradiction. $\square$

The upshot is that compactly supported distributions are the "dual" of $C^\infty(\mathbb{R}^n)$.

Tempered Distributions

REcall that $S(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) | \sup_{x \in \mathbb{R}^n} (1 + |x|^n) |\partial^\alpha \phi|(x) < \infty \forall n \geq 1, \alpha\}$.

This also comes with a Frechet Space structure, by $\phi \mapsto \sup_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} (1 + |x|^n) |\partial^\alpha \phi|(x)$.

The dual of $S(\mathbb{R}^n)$, when restricted to $C_0^\infty(\mathbb{R}^n)$, is the Tempered Distributions.

What is an application of all this?

An important application is fundamental solutions for constant coefficient PDEs.

As a technical tool, we define a convolution of distributions. If ϕ, ψ functions and $\phi * \psi(x) = \int_{\mathbb{R}^n} \phi(x-y), \psi(y)dy = \langle \psi_y, \phi(x-y)\rangle$. We generalize this to the case where ψ is a distribution, if $u \in D'(\mathbb{R}^n)$, and $\phi \in C_0^\infty(\mathbb{R}^n)$, then $u * \phi = u_y(\phi(x-y))$.

Exercise 17.1. *Check that $u_y(\phi(x-y))$ is a smooth function of x. Also, it is compactly supported if u is.*

Now, we can define the convolution to two distributions, u_1, u_2 if u_2 is compactly supported to be $(u_1 * u_2) * \phi = u_1 * (u_2 * \phi)$ for $\phi \in C_0^\infty(\mathbb{R}^n)$. This defines it uniquely!

Definition 17.2. *A constant coefficient linear differential operator on $\mathbb{R}^n$ is a finite linear combination $P = \sum a_\alpha \partial^\alpha$.*

Definition 17.3. *A distribution E is called a fundamental solution for P provided that $PE = \delta_0$.*

Use of this: consider the problem $Pu = f$.

Claim: $E * f$ solves this. $Pu = P(E*f) = (PE)*f = \delta_0 * f = f$. This is also called Green's Functions in special cases (like the Laplacian).

Important examples:

Laplace equation on $\mathbb{R}^n$ and heat equation on $\mathbb{R}^{n+1}$ (n spacial dimensions)

Theorem 17.1. *Set $E(x) = (2\pi)^{-1}\log|x|$ for $x \in \mathbb{R}^2\backslash\{0\}$, $E(x) = \frac{-|x|^{2-n}}{n-2}cu^{-1}$, $x \in \mathbb{R}^n \setminus \{0\}$ for $n \geq 3$ and c_n the volume of the unit sphere in $\mathbb{R}^n$. Then $\partial_j E$ in the dense of distributions given by $\frac{x_j|x|^{-n}}{c_n}$, $\Delta E = \delta_0$.*

Proof. Note that $E(x) \in L^1_{loc}(\mathbb{R}^n)$, $E(x) \in D'(\mathbb{R}^n)$, $\frac{\partial}{\partial x_j}E \in L^1_{loc}(\mathbb{R}^n)$.

We use the Divergence Theorem, which is $\int_S \vec{v}(x)\cdot \vec{n} dS = \int_D \div \vec{V} dx$.

Let $\phi \in C_0^\infty(\mathbb{R}^n)$, $\langle \partial_j E, \phi\rangle = -\langle E, \partial_j\phi\rangle = -\lim_{\epsilon\to 0}\int_{|x|>\epsilon} E(x)\partial_j\phi(x)dx = \lim_{\epsilon\to 0}\int_{|x|>\epsilon}\phi(x)\partial_j E(x)dx + \lim_{\epsilon\to 0}\int_{|x|=\epsilon} E(x)\frac{x_j}{|x|\cdot\vec{n}}\phi(x)dx$. The first term there ebcomes $\int \phi(x)\partial_j E(x)dx$, and we must compute the second term.

$\left|\int_{|x|=\epsilon} E(x)\langle x_i\rangle/|x|\vec{n}\phi(x)dS|\right| \leq C\epsilon\log|\epsilon| \to 0$ as $\epsilon \to 0$. Hence, indeed, $\partial_j E = x_j|x|^{-n}/c_n$.

Now we must check that this is a fundamental solution.

We will do this by calculation. $\Delta E(x) = 0$ if $x \in \mathbb{R}^n \setminus \{0\}$. $\langle \Delta E, \phi\rangle = \langle E, \Delta\phi\rangle = \lim_{\epsilon\to 0}\int_{|x|>\epsilon}(E\delta\phi - \Delta E\phi)dx = \lim_{\epsilon\to 0}\int_{|x|>\epsilon}\div(E\operatorname{grad}\phi - \phi\operatorname{grad}E)dx$. By the divergence theorem, $\lim_{\epsilon\to 0}\int_{|x|=\epsilon}(\phi\operatorname{grad}E - E\operatorname{grad}\phi)\cdot\vec{n}dS$, and $\operatorname{grad}\phi \to 0$. Now $\operatorname{grad}E = -\frac{x_j|x|^{-n}}{c_n}$. Dot this with $\vec{n}$ and we get $\frac{1}{c_n}\epsilon^{-(n-1)}$, and so $\lim_{\epsilon\to 0}\int_{|x|=\epsilon}\phi\operatorname{grad}E\cdot\vec{n}dS = \phi(0)$, and this is the same as $\Delta E = \delta_0$. ☐

18 Lecture 18

Theorem 18.1. *The function $E(t,x) = (4\pi t)^{-n/2}\exp(-|x|^2/4t)$ for $t > 0$ and $E(x,t) = 0$ for $t \leq 0$ is a fundamental solution to $(\partial_t - \Delta)(E(x,t)) = \delta_0$, $\delta_0 \in D'(\mathbb{R}^{n+1})$.*

Proof. IF $|x| \neq 0$, then check that $E(t,x)$ is smooth and extends smoothly to $x = 0$ if $t > 0$. Then $\int_{\mathbb{R}^n}(4\pi t)^{-n/2}\exp(-|x|^2/4t)dx = 1$, and so $E(t,x) \in L^1_{loc}(\mathbb{R}^{n+1})$ ad hence is a distribution. When $t > 0$, $\frac{\partial E}{\partial x_j} = -x_j/2tE$ and $\Delta_x E = -nE/2t + |x|^2E/(4t^2) = \frac{\partial E}{\partial t}$. Thus, $\operatorname{supp}(E(t,x)) \subset \{0\}$.

$\langle(\partial_t - \Delta)E, \phi\rangle = -\langle E, \frac{\partial\phi}{\partial t} + \Delta_x\phi\rangle = \lim_{\epsilon\to 0}\int_{t>\epsilon} -E(t,x)(\frac{\partial\phi}{\partial t} + \Delta\phi)dxdt$. The divergence theorem then gives us $\lim_{\epsilon\to 0}\int E(\epsilon,x)\phi(\epsilon,x)dx = \lim_{\epsilon\to 0}\int_{\mathbb{R}^n} E(1,x)\phi(\epsilon,\sqrt{\epsilon}x)dx = \phi(0,0)$. ☐

Theorem 18.2. *Every constant coefficient partial linear differential operator $P \neq 0$ admits a fundamental solution $E \in D'(\mathbb{R}^n)$, $PE = \delta_0$.*

18.1 Sobolev Spaces

LEt $X \subset \mathbb{R}^n$ open. THen $L^p(X) \subset L^1_{loc}$ by Hölder, and $L^p(X) \subset D'(X)$. Let $u \in L^p(X)$, but $\partial^\alpha u \notin L^p(X)$.

Definition 18.1. *LEt $k \in \mathbb{N}$ and $W^{k,p}(X) = \{u \in D'(X) | \partial^\alpha u \in L^p(X) \forall |\alpha| \leq k\}$.*

These can be turned into Banach spaces by using the norm $\|u\|_{k,p,X} = \|u\|_{W^{k,p}(X)} = \left(\int \sum_{|\alpha|\leq k} |\partial^\alpha u|^p(x)dx\right)^{1/p}$

Lemma 18.3. *$W^{k,p}$ is a Banach Space.*

Proof. Completeness: LEt $\{u_j\}$ by a Cauchy Sequence. THen each $\partial^\alpha u_j \in L^p(X)$ is Cauchy. By the completeness of $L^p(X)$, $\partial^\alpha u_j \to u_\alpha$ in L^p for $|\alpha| \geq 1$ and for all $|\alpha| \leq k$, $u_j \to u$ in L^p.

Claim: $\partial^\alpha u = u_\alpha$.

$\langle \partial^\alpha um, \phi) = (-1)^{|\alpha|}\langle u, \partial^\alpha\rangle = \lim_{j\to\infty}(-1)^{|\alpha|}\langle u_j, \partial^\alpha\phi\rangle = \lim_{j\to\infty}\langle\partial^\alpha u_j, \phi\rangle = \langle u_\alpha, \phi\rangle \Rightarrow \partial^\alpha u = u_\alpha$ in $D'(\mathbb{R}^n)$. ☐

Corollary 18.4. *If we embed $W^{k,p} \subset \prod_{|\alpha|\leq k} L^p(X)$ by $u \mapsto (\partial^\alpha u)_{|\alpha|\leq k}$, then the iomage is closed. Hence if $1 < p < \infty$ then $W^{k,p}$ is reflexive.*

If $X = \mathbb{R}^n$, and $p = 2$, then we can use the alternative definition via Plancherel $\|u\|_{L^2(\mathbb{R}^n)} = \|\hat{u}\|_{L^2(\mathbb{R}^n)}$.

$u \in W^{k,2}(\mathbb{R}^n)$ if and only if $\int_{\mathbb{R}^n}(1 + |\xi|^2)^k|\hat{u}(\xi)|^2 d\xi < \infty$.

Call $W^{k,2} = H^k$, and check that $\|u\|_{W^{k,2}} = \|u\|_{H^k}$.

Sobolev Embedding

Definition 18.2. *$W_0^{k,p}(X)$ is the closure of $C_0^k(X)$ with respect to $\|\cdot\|_{W^{k,p}(X)}$.*

Remark 18.1. *If $X = \mathbb{R}^n$, then $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$. These are essentially the functions with zero boundary value.*

Theorem 18.5 (Sobolev Embedding Theorem)**.** *Let $X \subset \mathbb{R}^n$ open, then $W_0^{1,p}$ embeds into $C^{np/(n-p)}(X)$ if $p < n$ and into $L^0(X)$ if $p > n$.*

Moreover, we have that $\|u\|_{np/(n-p)} \leq C(p,n)\|\nabla u\|_{L^p}$, $p < n$ and $\sup_X |u| \leq C(u,x)\|\nabla u\|_{L^p}$ for $p > n$.

Remark 18.2. *The reason for the $np/(n-p)$ exponend is scaling comptibility. If we take $u(x)$ to $u(\lambda x)$ for $\lambda \neq 0$, then $\|u(\lambda x)\|_{L^{np/(n-p)}} = \frac{1}{(\lambda^n)^{(n-p)/np}}\|u\|_{L^{np/(n-p)}}$, which means that scaling this gives a multiplication by $\lambda^{-p/(n-p)}$. Scaling ∇u gives the opposite.*

Proof. Our choice of $W_0^{1,p}$ means that it suffices to consider $u \in C_0^1(X)$ using density argument.

First consider $p = 1$. Then $|u(x)| \leq \int_{-\infty}^{x_i} \left|\frac{\partial}{\partial x_i} u\right| dx_i$ for $i = 1, \ldots, n$. Thus $|u(x)|^{n/(n-1)} \leq \left(\prod_{i=1}^n \int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x_i}\right| dx_i\right)^{1/(n-1)}$.

Now we apply Hölder successively in each variable:

$$\begin{aligned}
\|u\|_{L^{n/(n-1)}} &= \left(\int_{\mathbb{R}^n} |u(x)|^{n/(n-1)} dx\right)^{(n-1)/n} \\
&\leq \left[\int_{\mathbb{R}^n} \left(\prod_{i=1}^n \int_{-\infty}^{\infty} \left|\frac{\partial}{\partial x_i} u\right| dx_i\right)^{1/(n-1)}\right]^{(n-1)/n} \\
&\leq \left[\prod_{i=1}^n \left(\int \left|\frac{\partial u}{\partial x_i}\right| dx_1 dx_2 \ldots dx_n\right)^{1/(n-1)}\right]^{(n-1)/n} \\
&\leq \|\nabla u\|_{L^1}
\end{aligned}$$

So this settles the case where $p = 1$.

For the case $p > 1$, replace u by $|u|^\gamma$ where $\gamma > 1$ is to be chosen. Then what we've proved gives us $\||u|^\gamma\|_{n/(n-p)} \leq \gamma \int |u|^{\gamma-1}|\nabla u| dx \leq \gamma \||u|^{\gamma-1}\|_{L^p}\|\nabla u\|_{L^p}$.

So $\gamma n/(n-1) = (\gamma-1)p/(p-1)$, and so $\gamma = (n-1)p/(n-p)$. Then the inequality implies that $\|u\|^\gamma_{np/(n-p)} \leq \gamma\|u\|^{\gamma-1}_{np/(n-p)}\|\nabla u\|_{L^p}$, and so $\|u\|_{np/(n-p)} \leq \gamma\|\nabla u\|_{L^p}$.

Now we consider the case $p > n$. Then use a boot strap technique. $u \in C_0^1(X)$, and also assume that $m(X) = 1$. Then define $\tilde{u} = \frac{|u|}{\|\nabla u\|_{L^p(X)}}$.

By the preceding steps, $\|\tilde{u}^\gamma\| \leq \gamma\|\tilde{u}^{\gamma-1}\|_{L^{p/(p-1)}} \leq \gamma^{1/\gamma}\|\tilde{u}\|_{p'\gamma} \geq \gamma^{1/\gamma}\|\tilde{u}\|^{1-1/\gamma}_{p'(1-\gamma)} \geq \|\tilde{u}\|_{n'\gamma}$.

Now put $\gamma = \delta^k$ for $k = 1, \ldots$ and $\delta = n'/p' > 1$.

$\|\tilde{u}\|_{n'\delta^k} \leq \delta^{k\delta^{-k}}\|\tilde{u}\|^{1-\delta^{-k}}_{n'\delta^{k-1}} \leq \delta^{k\delta^{-k}}\delta^{(k-1)\delta^{-(k-1)}(1-\delta^{-k})}\|\tilde{u}\|^{(1-\delta^{-k})(1-\delta^{-k-1})}_{n'\delta^{k-2}}$.

Iterating, this is less than or equal to $\delta^{\sum_{k\geq 1} k\delta^{-k}} \|\tilde{u}\|_{n'} \leq M\|\nabla\tilde{u}\|_{L^1} \leq M\|\nabla\tilde{u}\|_{L^p} \leq M = \delta^{\sum k\delta^{-k}}$.

The proof is concluded by the following lemma: □

Lemma 18.6. *Let $X \subset \mathbb{R}^n$ bounded and u measurable. Then* $\operatorname{esstlsup}_X |u| \leq \lim_{p\to\infty} \left(\frac{1}{|X|}\int_X |u|^p dx\right)^{1/p}$.

In particular, the limit exists in $[0,\infty]$.

Recall that $\operatorname{esstlsup}|u| = \int_{N\subset X, |N|=0} \sup_{X\setminus N} |u|$.

19 Lecture 19

19.1 Spectral Theory of Operators on Hilbert Space

Let $\mathscr{H}$ be Hilbert Space and $T : \mathscr{H} \to \mathscr{H}$ bounded linear map. Then we can associate an adjoint with T

Proposition 19.1. *There exists a unique continuous linear map $T^* : \mathscr{H} \to \mathscr{H}$ such that*

1. *$(Tf, g) = (f, T^*g)$ for all $f, g \in H$.*
2. *$\|T\| = \|T^*\|$*
3. *$(T^*)^* = T$.*

One calls T^* the adjoint of T.

Proof. Fix $g \in H$. Then define a linear functional $g : H \to \mathbb{C}$ by $g(f) = (Tf, g)$. This is bounded, and so by Riesz Representation Theorem, there exists an element named T^*g such that $g(f) = (f, T^*g)$. Then clearly T^*g depends linearly on g.

$\|g\| = \sup\{|(Tf, g)|, \|f\| \leq 1, \|g\| \leq 1\} = \sup\{|(f, T^*g)|, \|f\| \leq 1, \|g\| \leq 1\} = \|T^*\|$. And finally 3 follows by conjugation. □

Definition 19.1 (Symmetric)**.** *A continuous linear operator $T : H \to H$ is called symmetric (self-adjoint) if $T^* = T$.*

Remark 19.1. *In general, symmetric and self-adjoint are not the same.*

Definition 19.2 (Compact)**.** *A continuous linear operator $T : H \to H$ is called compact if $\overline{T(B_1(0))} \subset H$ is compact.*

Theorem 19.2 (Spectral Theorem for Compact Symmetric Operators)**.** *Let $T : H \to H$ be compact symmetric. Then there exists an orthonormal basis $\{\phi_k\}$ of H consisting of eigenvectors for T, that is, $T\phi_k = \lambda_k\phi_k$ for $\lambda_k \in \mathbb{R}$ and $H = \overline{\operatorname{span}\{\phi_k\}}$.*

Furthermore, $\lambda_k \to 0$ as $k \to \infty$. If $\lambda \neq 0$, then the dimension of the eigenspace E_λ is finite.

Remark 19.2. *The numbers $\{\lambda_k\}$ are called the spectrum of T. If $\lambda \neq 0$ and $\lambda \notin \{\lambda_k\}$, then $(T - \lambda I)^{-1}$ is bounded on E_k for each k and is given by $(-\lambda + \lambda_k)^{-1}$.*

Hence $\|(T - \lambda I)^{-1}\|_H = \sup\{(\lambda - \lambda_k)^{-1}\} = \max\{(\lambda - \lambda_k)^{-1}\} < \infty$

Lemma 19.3. *Let T be as in the theorem. Then*

1. *If λ is an eigenvalue, then $\lambda \in \mathbb{R}$.*
2. *If f_1, f_2 are eigenvectors belonging to different eigenvalues, then $(f_1, f_2) = 0$.*

Proof.

1. If $Tf = \lambda f$, ten $\lambda(f, f) = (Tf, f) = (f, Tf) = \bar{\lambda}(f, f)$, and since $(f, f) \neq 0$, $\lambda = \bar{\lambda}$
2. If $Tf_i = \lambda_i f_i$ for $i = 1, 2$ and $\lambda_1 \neq \lambda_2$, then $\lambda_1(f_1, f_2) = (Tf_1, f_2) = (f_1, Tf_2) = \lambda_2(f_1, f_2)$, and so $(f_1, f_2) = 0$.

$\square$

Lemma 19.4. *Same assumptions as in theorem. Then for $\lambda \neq 0$, the eigensapce E_λ is finite dimensional. The eigenvalues of T for an at most denumerable set.*

Proof. For first assertion, assume E_λ is infinite dimensional. Then there exists a countably infinite orthonormal set $\{\phi_k\} \subset E_\lambda$ with $T\phi_k = \lambda\phi_k$. By compactness, there exists a subsequence $T\phi_{k_n}$ which converges. This can't be, as $\|\phi_{k_n} - \phi_{k_m}\|^2 = \|\phi_{k_n}\|^2 + \|\phi_{k_m}\|^2 = 2$.

For second assertion, we'll show that for $\mu > 0$, there are only finitely many eigenvalues λ with $|\lambda| \geq \mu$. Then the eigenvalues are the union of the ones greater than $\frac{1}{n}$, and so are denumerable.

Assume that there are infinitely many λ with $|\lambda| \geq \mu$. Then we choose at least countably many of their eigenvectors $\{\phi_k\}$ with $T\phi_k = \lambda_k\phi_k$ orthonormal.

Then a subsequence $T\phi_{k_n}$ converges, $\|\lambda_{k_n}\phi_{k_n} - \lambda_{k_m}\phi_{k_m}\|^2 = |\lambda_{k_n}|^2 + |\lambda_{k_m}|^2 \geq 2\mu^2$, contradiction. $\square$

Lemma 19.5 (Existence of Eigenvalues)**.** *Same assumptions. Then either $\|T\|$ or $-\|T\|$ is an eigenvalue.*

Proof. Claim: $\|T\| = \sup_{\|f\|=1} |(Tf, f)|$. By Cauchy-Schwartz, $|(Tf, f)| \leq \|T\|$. Conversely, we use the following algebriac trick: $(Tf, g) = \frac{1}{4}[(T(f+g), f+g) - (T(f-g), f-g) + i(T(f -$ and so $(Tf, f) = (f, Tf) = \overline{(Tf, f)}$, and so $(Tf, f) \in \mathbb{R}$. Thus $\Re(Tf, g) = \frac{1}{4}[(T(f+g), f+g) - (T(f-g), f-g)]$

And so $|\Re(Tf, g)| \leq \frac{1}{4}\sup_{\|f\|\leq 1} |(Tf, f)|[\|f+g\|^2 + \|f-g\|^2]$. And so $\sup_{\|f\|\leq 1, \|g\|\leq 1} |\Re(Tf, g)| \leq \sup_{\|f\|\leq 1} |(Tf, f)|$. To get rid of $\Re$, for arbitrary $f, g \in H$ with $\|f\|, \|g\| \leq 1$, choose $\theta \in \mathbb{R}$ such that $(T(e^{i\theta}f), g) \in \mathbb{R}$.

Then $|(Tf, g)| = |e^{-i\theta}(T(e^{i\theta}f), g)| = |(T(e^{i\theta}f), g)| = |\Re(T(e^{i\theta}f), g)| \leq \sup_{\|f\|\leq 1} |(Tf, f)|$, and so the claim is established.

Now we know that $\|T\| = \sup_{\|f\|=1} |(Tf, f)|$, and so either $\|T\| = \sup_{\|f\|=1}(Tf, f)$ or $-\|T\| = \inf_{\|f\|=1}(Tf, f)$ or both.

Assume the first case. Pick a sequence $\{f_n\} \subset H$ with $(Tf_n, f_n) \to \|T\|$ and $\|f_n\| = 1$. By compactness, we can choose a subsequence (which we will again label f_n) such that $Tf_n \to g \in H$. We claim that g is an eigenvector with eigenvalue $\|T\|$.

$\|Tf_n - \|T\|f_n| \to 0$. Indeed, $\|Tf_n - \|T\|f_n\|^2 = \|Tf_n\|^2 + \|T\|^2 - 2\|T\|(Tf_n, f_n) \to 0$. Since $Tf_n \to g$, then $\|T\|f_n \to g$, and so $\|T\|Tf_n \to \|T\|g$ and to Tg, and so $Tg = \|T\|g$. □

We now finish the proof of the Spectral Theorem.

Proof. Denote the closure of the span of the eigenvectors of T on H by S, $S \neq \{0\}$. We calim that $S = H$. If not, then $H = S \oplus S^\perp$. Check that $T : S \to S$ and $S^\perp \to S^\perp$. Then by the last lemma, there exists an eigenvector $v \in S^\perp$, contradiction. □

Example 19.1 (Hilbert-Schmidt Operators)**.** $H = L^2(\mathbb{R}^n)$, $Tf = \int_{\mathbb{R}^n} k(x,y)f(y)dy$ *and* $k(x,y) \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ *is the Hilbert-Schmidt Kernel.*

Theorem 19.6. *T is a compact operator on $L^2(\mathbb{R}^n)$. Its adjoint is also Hilbert-Schmidt, with kernel $\overline{k(x,y)}$.*

Remark 19.3. *In particular, if $k(x,y) = \overline{k(x,y)}$, then the spectral theorem applies.*

Proof. First note that $\int k(x,y)f(y)dy$ is well defined for almost every $x \in \mathbb{R}^n$ by Fubini's theorem. By C-S, $k(x,y)f(y) \in L^1(\mathbb{R}^n)$ for almost every x.

C-S says that $\int k(x,y)f(y)dy \leq \left(\int |k(x,y)|^2 dy\right)^{1/2} \left(\int |f(x)|^2 dy\right)^{1/2} \leq \iint |k(x,y)^2 dydx \|f\|_{L^1}^2 \leq C\|f\|_{L^1}^2$.

Now we check compactness. Choose an orthonormal basis $\{\phi_n\}$ for $L^2(\mathbb{R}^n)$. Then the set of functions $\{\phi_n(x)\phi_m(y)\}$ on $\mathbb{R}^n \times \mathbb{R}^n$ is orthonormal. We calim it's a basis for $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. To see this, assume that $(g(x,y), \phi_n(x)\phi_n(y)) = 0$ for all n, m.

Then $\left(\int g(x,y)\phi_m(y)dy, \phi_n x)\right) = 0$, so $\int g(x,y)\phi_m(y)dy = 0$ for almost every x, and so $g(x,y) = 0$ for almost every x, y. So $k(x,y) = \sum_{n,m\geq 1} a_{nm}\phi_n(x)\phi_m(y)$ where $a_{nm} = (k(x,y), \phi_n(x)\phi_m(y))$.

Now we define the operator T_ℓ for $\ell \geq 1$. $T_\ell f = \int k_\ell(x,y)f(y)dy$, where $k_\ell(x,y) = \sum_{n,m=1}^{\ell} a_{nm}\phi_n(x)\phi_m(y)$. Note that the image of T_ℓ has dimension $\ell < \infty$. T_ℓ is compact. Further, $T_\ell \to T$.

Lemma 19.7. *Assume $T_\ell : H \to H$ is compact and $T_\ell \to T$ as $\ell \to \infty$. Then T is compact.*

Proof. Given $\{f_n\} \subset H$ with $\|f_n\| \leq 1$, choose a subsequence $\{f_m\}$ such that $T_1 f_m$ converges. Then choose a subsequence f_{2m} such that $T_2 f_{2n}$ converges. Continue interatively. Let g_n be the diagonal sequence.

Claim: $\{Tg_n\}$ converges. Given $\epsilon > 0$, choose ℓ large enough such that $\|T - T_\ell\| < \epsilon/3$. Then choose k large enough such that $\|T_\ell g_n - T_\ell g_m\| < \epsilon/3$ for all $n, m > k$. Then $\|Tg_n - Tg_m\| \leq \|T_\ell g_n - T_\ell g_m\| + \|(T - T_\ell)g_n\| + \|(T - T_\ell)g_m\| \leq \epsilon$ and so we are done. □

□

20 Lecture 20

$\Delta u = f$, Δ on a bounded domain $\Omega \subset \mathbb{R}^n$. $L^2(\Omega) = \oplus E_k$, wieth $\Delta|_{E_k} = \lambda_k I_{E_k}$ for $E_k \subset C^\infty(\Omega)$.

Theorem 20.1 (Rellich-Kondrakor Compactness Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ a bounded open set. Then $W_0^{1,p}(\Omega)$ embeds compactly into $L^q(\Omega)$ for any $q < \frac{np}{n-p}$ if $p < n$.*

Proof. First we establish the compactness of the embedding $W_0^{1,p}(\Omega) \subset L^1(\Omega)$. LEt $A \subset W_0^{1,p} \subset W_0^{1,p}(\Omega)$ bounded by scaling, we may assume that $\|u\|_{W_0^{1,p}(\Omega)} \leq 1$ for all $u \in A$.

By using density, we may assume that $A \subset C_0^1(\Omega)$.

Let $\rho \geq 0$, $\rho \in c_0^\infty(B_1(0))$, $\int_{\mathbb{R}^n} \rho(x)dx = 1$. $u_h = \int_{\mathbb{R}^n} \rho_n(x-y)u(y)dy$, $\rho_n(x) = h^{-n}\rho(x/h)$, $h > 0$. And so $A_h = \{u_h | u \in A\}$.

$|u_h(x)| \leq Ch^{-n}$, $C = C(\sup_{u \in A} \|u\|_{W_0^{1,p}}, \Omega)$, $|\nabla u(x)| = |\int \nabla \rho_h(x-y)u(y)dy| \leq Ch^{-n-1}$, and so we now use the Arzela-Ascoli theorem to see that $\bar{A}_h$ is compact in $C^0(\bar{\Omega})$, and so $\bar{A}_h$ is compact in $L^1(\Omega)$.

Theorem 20.2 (Arzela-Ascoli)**.** *Let $K \subset \mathbb{R}^n$ compact, tehn a closed subset $A \subset C^0(K)$ is compact if and only if it is bounded and equicontinuous.*

To conclude, we need to compare u_1, u_h. $|u(x) - u_h(x)| \leq \int_{|z|\leq 1} \rho(z)|u(x) - u(x-hz)|dz = \int_{|z|\leq 1} \rho(z)|\int_0^{h|z|} D_r u(x-rw)|drdz$ where $w = z/|z|$.

And so, $\int |u(x) - u_h(x)|dx \leq \int \int_{|z|\leq 1} \rho(z)|\int_0^{h|z|} D_r u(x-r\omega)|drdzdx \leq \int_{|z|\leq 1} \rho(z) \int_0^{h|z|} \int_\Omega D_r u(x-r\omega)|dxdrdz \leq \|\nabla u\|_{L^1} h \leq \|\nabla u\|_{L^p} hC(\Omega)$.

This implies compactness of A: let $\{x_n\} \subset A$ be given. Tjem cjppse $\{u_{1n}\}$ a subsequence such that it converges in $L^1(\Omega)$. Then choose a subseqeunce u_{2n} which converges in L^1(???) Then inductively choose u_{in} that converges. Then consider u_{nn}, then this converges in $L^1(\Omega)$. And so $W_0^{1,p} \subset L^1(\Omega)$ compactly. In fact, this is to $L^q(\Omega)$, as $\|u\|_{L^q}^q = \int_\Omega |u|^q dx = \int_\Omega |u|^a |u|^b dx$ for $a + b = q$.

$\int_\Omega |u|^a|u|^b dx \leq (\int_\Omega |u|^{ap_1}dx)^{1/p_1}(\int_\Omega |u|^{bp_2}dx)^{1/p_2}$ where $1/p_1 + 1/p_2 = 1$. If $ap_1 = 1$ and $bp_2 = \frac{np}{n-p}$, and denote $a = \lambda$, we get $\frac{q-\lambda}{1-\lambda} = \frac{np}{n-p}$ and we can solve this for λ.

And so if $\{u_{nn}\}$ converges in $L^1(\Omega)$, then $\|u_n - u_m\|_{L^q} \leq \|u_n - u_m\|_{L^1(\Omega)}^{a/q} \|u_n - u_m\|$, and so $\{u_n\}$ converges in $L^q(\Omega)$. □

Note now that $\Delta u = f$, everything on a bounded domain and $u \in W_0^{1,2}(\Omega)$, then we can interpret this equation weakly!

$\partial_i(\partial_i f) = f$ summed over the repeated index, and this holds if and only if $(\partial_i u, \partial_i v) = (f, v)$ for all $v \in W_0^{1,2}(\Omega)$.

$\mathscr{L}(u,v) = -(\partial_i u, \partial_i v)$ is a bilinear form on $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$. Define $\mathscr{L}_\delta(u,v) = -(\partial_i u, \partial_i v) - \delta(u,v)$ for $\delta > 0$ arbitrary.

Terminology: Let H over $\mathbb{R}$ be a Hilbert Space and $B : H \times H \to \mathbb{R}$ bilinear. We call B coercice provided that $B(u,u) \geq \lambda \|u\|^2$ for some $\lambda > 0$.

Clearly, $\mathscr{L}_\delta(u,u) \geq \min\{\delta, 1\} \|u\|^2_{W_0^{1,2}(\Omega)}$. General fact about coercive linear forms:

Theorem 20.3 (Lax-Milgram)**.** *Let $B : H \times H \to \mathbb{R}$ bounded, coercice and bilinear. Then for every $F \in H^*$, there exists a unique element $f \in H$ such that $B(x,f) = F(x)$ for all $x \in H$.*

Proof. By Riesz Representation Theorem, there exists a map $T : H \to H$ such that $B(x,f) = (x, Tf)$, with T linear and bounded. $B(Tf, f) = (Tf, Tf) \leq C\|Tf\|\|f\|$, and so $C\|f\| \geq \|Tf\|$.

$\lambda \|f\|^2 \leq B(f,f) = (f, Tf) \leq \|f\|\|Tf\|$, and so $\lambda\|f\| \leq \|Tf\|$ and so T is 1-1, and T^{-1} is bounded on the rage, and T has closed range. We need to show that T is onto.

Assume that T is not onto. By the closedness of the image, there exists $z \neq 0$, $(z, Tf) = 0$ for all $f \in H$. Then put $f = z$, and so $(z, Tz) = 0$ and so $B(z,z) = 0$, by coercivity, $z = 0$, contradiction. $\square$

$\mathscr{L}_\delta$ is coercive and bounded on $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$. $h_\delta^{-1} : (W_0^{1,2}(\Omega))^* \to W_0^{1,2}(\Omega)$, $F(u) = \mathscr{L}_\delta(u, h_\delta^{-1}\mathscr{F})$ for all $F \in (W_0^{1,2}(\Omega))^*$.

$L^2(\Omega) \subset (W_0^{1,2}(\Omega))^*$: if $v \in L^2(\Omega)$, then $v(u) = (u,v)_{L^2}$ for $u \in W_0^{1,2}$.

And so $h_0^{-1} : L^2 \to W_0^{1,2} \to L^2$.

If $K = i^{L^2}_{W_0^{1,2}} \circ h_\delta^{-1}$, where i is the inclusion $W_0^{1,2} \subset L^2$, then we must check that K is symmetric and compact. Then the spectral theorem applies, and so there is a countable set $\{\lambda_k\}$ oand corresponding finite dimensional spaces $E_k \subset L^2$ such that $h_\delta^{-1}|_{E_k} = \lambda_k \operatorname{id}|_{E_k}$.

And so $L^2 = \oplus E_k$, and to translate back to the level of the Laplacian, $h_0^{-1}\phi = \lambda\phi$ implis that $\mathscr{L}_\delta(v,\phi) = (v,\phi)$, and so $-\lambda \int_\Omega \partial_i v \partial_i \phi dx - \lambda\delta \int v\phi dx = (v,\phi)$, if and only if $(\Delta - \delta)\phi = \lambda^{-1}\phi$ in the weak sense, and so $\Delta\phi = (\delta + \lambda^{-1})\phi$.

21 Lecture 21

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22 Lecture 22

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23 Lecture 23

Schrodinger Equation $-\Delta\psi + V\psi(x) = E\psi$. We will construct the ground state solution.

$\epsilon(\psi) = \int_{\mathbb{R}^n} |\nabla\psi|^2(x)dx + \int V(x)|\psi|^2 dx$.

Theorem 23.1. *If $V \in L^{n/2} + L^{\infty}$, for all $a > 0$, $|\{|V(x)| > a\}| < \infty$ and $\int \epsilon(\psi) = E_1 < 0$ on the unit sphere, then there exists a minimzer $\psi_0 \in H$ for $\epsilon(\psi)$ which satisfies the Schrodinger Equation, in the distributional sense.*

Further, if $V \in C^{\infty}(\mathbb{R}^n)$, then so is ψ_0.

Strategy:

1. Show coercivity: that is, $\int |\nabla\psi|^2 dx \leq C\epsilon(\psi) + D\|\psi\|_{L^2}^2$ where C, D depend on V, and here we only need that $V \in L^{n/2} + L^{\infty}$.
2. SHow the weak continuity of $\epsilon(\psi)$.
3. Use the first two parts to construct ψ_0.

Part 1 was last time.

For part 2, we need that

Proposition 23.2. *Let V be as in the main theorem. Then if $\psi_j \to \psi$ in H, then $\int V(x)|\psi_j|^2 dx \to \int V(x)|\psi|^2 dx$.*

Proof. Replace V with V^{δ} which is bounded, by defining $V^{\delta}(x) = \begin{cases} V(x) & |V(x)| < 1/\delta \\ 0 & else \end{cases}$.

Then replace $\mathbb{R}^n$ by a set A of bounded measure. Here we use the fact that $|\{|v|(x) > a\}| < \infty$ for all $a > 0$. We reduced to the following lemma: $\square$

Lemma 23.3 (Weak Convergence implies Strong Convergence)**.** *Let $\psi_j \in H'(\mathbb{R}^n)$ such that $\nabla\psi_j \to v$. Then $v = \nabla\psi$ for some suitable ψ in $H'(\mathbb{R}^n)$, and for every $A \subset \mathbb{P}^n$ of bounded measure, $\chi_A\psi_j \to \chi_A\psi$ with respect to L^p for $2 \leq p \leq \frac{2n}{n-2}$.*

Proof. By the principle of uniform boundedness, $\|\nabla\psi_j\|_{L^2} \leq C$ for all j. We may assume that $C = 1$. By Sobolev embedding, $\|\psi_j\|_{L^{2n/n-2}} \leq C$ for all j. By reflexivity, there is a subsequence $\{\psi_{k_j}\}$ such that $\psi_{k_j} \to \psi$, with $\|\psi_{k_j}\| \leq D$, and so $\|\psi\| \leq D$.

We claim that the full sequence $\psi_j \to \psi$. If not, then some other subsequence goes to $\tilde{\psi} \neq \psi$.

$\phi \in C_0^{\infty}(\mathbb{R}^n)$, $-\int \psi\partial_i\phi dx = \lim_{j\to\infty} -\int \psi_{k_j}\partial_i\phi dx = \lim_{j\to\infty}\int \partial_i\psi_{k_j}\phi dx$, and similary for $\tilde{\psi}$. And so everything is equal, and so $\psi = \tilde{\psi}$.

Furthermore, $\nabla\psi = v$. This establishes the first part of the lemma. Now we use this to show strong local convergence. $\psi_j \to \psi$. The idea is to regularize the ψ_j. $\psi_j \mapsto \tilde{\psi_j} = e^{t\Delta}\psi_j(x) = (4\pi t)^{-n/2}\int_{\mathbb{R}^n} e^{-|x-y|^2/(4t)}\psi_j(y)dy$.

So now $\|\psi_j - e^{t\Delta}\psi_j\|^2 = \int |\hat{\psi_j}|^2(\xi)(1 - e^{-4\pi|\xi|^2 t})^2 d\xi$.

Some inequalities follow.

I'm lost. $\square$

And so now we finish the proof of the main theorem.

Choose a minimizing sequence $\psi_j \in H'$ with $\|\psi_j\| = 1$. That is, $\epsilon(\psi_j) \to \int \epsilon(\psi) = E_0$.

By part 1, $\int |\nabla\psi_j|^2 dx \leq C\epsilon(\psi_j) + D\|\psi_j\|^2 = C\epsilon(\psi_j) + D$, and so $\|\psi_j\| \leq M$ in H' norm.

By reflexivity of H', a subsequence again denoted by ψ_j converges weakly to some ψ_0. $\nabla\psi_j \to v$, and so $\int V(x)|\psi_j|dx \to \int V(x)|\psi_0|^2 dx$, and $\int |\Delta\psi_j|^2 dx$ is decreasing.

Thus, $\epsilon(\psi_0) \leq \liminf_j \epsilon(\psi_j) = E_0$. So $\|\psi_0\|_{L^2} \leq 1$, so $E_0 \geq \epsilon(\psi_0) \geq \|\psi_0\|_{L^2} E_0$, by definition, and so $\|\psi_0\| = 1$ and $\epsilon(\psi_0) = E_0$, and so $\psi_0 \neq 0$ works. We must just show that it satisfies Schrodinger in the distributional sense.

$\psi_0 + \epsilon\eta$ for $\eta \in C_0^\infty(\mathbb{R}^n$ and $\epsilon \in \mathbb{R}$.

Let $L_\epsilon = \epsilon(\psi_0 + \epsilon\eta) \geq \epsilon(\psi_0)$ (note, all things are normalized). This holds for all ϵ, η.

Then $\frac{dL}{d\epsilon}|_{\epsilon=0} = 0$, and so (mess inequalities and equations) we get the result, that $-\Delta\psi_0 + V(x)\psi_0 = E_0\psi_0$ in $D'(\mathbb{R}^n)$.

In addition to all the requirements, we also assume that V is smooth of compact support. Claim is that $\psi_0 \in C^\infty(\mathbb{R}^n)$. Then we claim that $(-\Delta - E_0)\psi_0 = V\psi_0 \in H'$.

$\psi_0 = (-\Delta - E_0)^{-1}(V\psi_0)$, which is in H'. So then we have ...gibberish.

I don't understand this. I'm fucked on Thursday.

24 Lecture 24

25 Lecture 25