Recommended Books: ???

1 Lecture 2

Math 618 Get Ahead class: tomorrow 1:30-3:00 in DRL 4C4

Two theorems we will prove:

Theorem 1.1 (Whitehead Theorem). *If $f : X \to Y$ is a pointed morphism of CW Complexes such that $f_* : \pi_k(X, x) \to \pi_k(Y, f(x))$ is an isomorphism for all k, then f is a homotopy equivalence.*

Example 1.1. *$C \subset [0, 1]$ the Cantor Set. Let C^δ be the Cantor set with the discrete topology. Then $C^\delta \to C$ induces isomorphisms on all homotopy groups, but it is not a homotopy equivalence, so the CW hypothesis is required.*

Theorem 1.2 (Hurewicz Theorem). *Let X be a space and $\pi_k(X, x) = 0$ for $k < n$. Then $H_k(X, x_0) = 0$ for $k < n$ and $\pi_n(X, x_0) \simeq H_n(X, x_0)$ for $n \geq 1$.*

Theorem 1.3 (Hurewicz Theorem, Relative Version). *Let (X, A) be a pair ($A \subset X$), then if $\pi_k(X, A) = 0$ for all $k < n$ then $H_k(X, A) = 0$ for $k < n$ and $\pi_n(X, A) \simeq H_n(X, A)$.*

Corollary 1.4 (Whitehead Theorem 2). *If X, Y are CW complexes, $\pi_1(X) = \pi_1(Y) = 0$ and $f : X \to Y$ induces an isomorphism on homology $f_* : H_k(X) \to H_k(Y)$ for all k, then f is a homotopy equivalence.*

Proof. Assume WLOG (which will be explained later) that $f : X \to Y$ is an inclusion. Then $\pi_1(Y, X) = 0$ because both are simply connected, and so $H_1(Y, X) = 0$.

And so, $\pi_2(Y, X) \simeq H_2(Y, X)$ by Hurewicz. And so now we look at a part of the long exact sequence on homology, $H_2(X) \to H_2(Y) \to H_2(Y, X) \to H_1(X) \to H_1(Y)$, and as $H_2(Y, X)$ is trapped between isomorphisms, it is zero, so $\pi_2(Y, X) = 0$.

So then $H_k(Y, X) = 0$ and so $\pi_k(Y, X) = 0$ for all k. So we have $0 \to \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \to \pi_k(Y, X) \to \pi_{k-1}(X) \to \pi_{k-1}(Y)$ so $\pi_k(Y, X) = 0$ for all k and so we get that $f_k : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for all k. Therefore, by Whitehead theorem 1, f is a homotopy equivalence. □

Application: Suppose that X^3 is a closed, simply connected 3-manifold. Then $X \simeq S^3$ (homotopy equivalent).

Proof. $\pi_1(X) = 0 \Rightarrow H_1(X) = 0 \Rightarrow H_2(X) = 0$ by Poincare Duality and Universal Coefficient Theorem, and $H_3(X) = \mathbb{Z}$, by Poincare Duality. Now, Hurewicz implies that $\pi_1(X) = \pi_2(X) = 0$ and $\pi_3(X) = \mathbb{Z}$. Obtain a map by $f : S^3 \to X$ mapping S^3 to the generator of π_3.

So now we claim that $f_* : H_*(S^3) \to H_*(X)$ is an isomorphism (easily checked), and so by Whitehead 2, f is a homotopy equivalence. □

Why is Homology easy to calculate and homotopy not?

Answer: Homology behaves well with respect to cellular inclusions (excision). Spaces like manifolds are easy to decompose by such inclusions. On the other hand, homotopy behaves badly with respect to these inclusions. There is a class of maps and spaces for which homotopy behaves well and homology behaves badly.

Definition 1.1 (Pair)**.** *A pair of topological spaces is just (X, A) where $A \subset X$. A map of pairs $f : (X, A) \to (Y, B)$ is a map $f : X \to Y$ such that $f(A) \subset B$. A product of pairs is $(X \times Y, X \times B \cup A \times Y)$. This is the Leibniz rule (try $A = \partial X$ and $B = \partial Y$ to see it)*

The category Top embeds into $2-Top$ (the category of pairs) by $X \mapsto (X, \emptyset)$. So $(X, A) \times (I, \emptyset) = (X \times I, A \times I)$.

Definition 1.2 (Homotopic Rel X')**.** *Let $f_0, f_1 : (X, A) \to (Y, B)$ are maps of pairs, $X' \subset X$, and $f_0|_{X'} = f_1|_{X'}$. Then we say that f_0 is homotopic to f_1 rel X' iff there exists a homotopy $H : (X, A) \times I \to (Y, B)$ such that $H(x, 0) = f_0(x)$, $H(x, 1) = f_1(x)$, $H(x, t) = f_0(x) = f_1(x)$ for $x \in X'$.*

Remarks/Easy Exercises:

1. $\simeq$ rel X' is an equivalence relation.
2. Composition of homotopic maps are homotopic. That is, if $f_0 \simeq f_1$ rel X' are maps from $(X, A) \to (Y, B)$ and $g_0 \simeq g_1$ rel Y' are maps $(Y, B) \to (Z, C)$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$ rel X'.

Notions of Homotopy Equivalence and Contractible work out here.

Definition 1.3 (Retract, Weak Retract)**.** *$A \subseteq X$ is called a retract if the inclusion has a left inverse $r : X \to A$ such that $ri = \mathrm{id}_A$.*

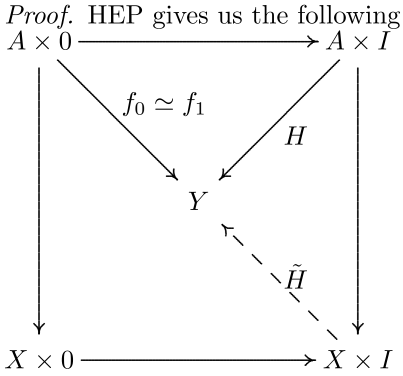
It's called a weak retract if there exists r such that $ri \simeq \mathrm{id}_A$.

Let A be the subset of I^2 which consists of the segments $(x, 0)$ for $x \in I$, $(0, y)$ for $y \in I$, and $(1/n, y)$ for $y \in I$. A is a weak retract, since $X \to A$ by $(x, y) \to (0, 1)$ collapses the whole thing in the y direction, then the x, then back up to the point. Then $A \to X \to A$ is homotopic to the identity, but A is not a retract of X.

Definition 1.4 (Homotopy Extension Property, HEP)**.** *A pair (X, A) has the homotopy extension property, (HEP), with respect to Y if given $f : X \to Y$, and $H : A \times I \to Y$ such that $H(a, 0) = f(a)$ for all $a \in A$, there exists $\tilde{H} : X \times I \to Y$ such that $\tilde{H}(x, 0) = f(x)$ for all $x \in X$ and $\tilde{H}(a, t) = H(a, t)$ for all $(a, t) \in A \times I$.*

Lemma 1.5. *Suppose that (X, A) has HEP relative to Y. Let $f_0, f_1 : A \to Y$ and $f_0 \simeq f_1$. Then if f_0 extends to X, then so does f_1.*

Proof. HEP gives us the following diagram. The result follows.

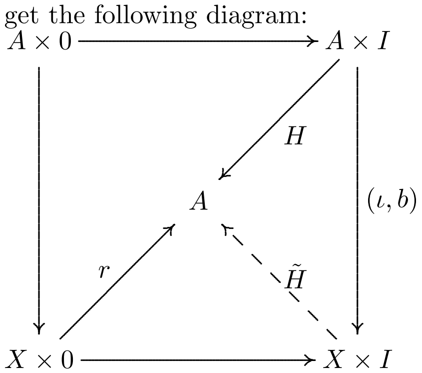


□

Definition 1.5 (Cofibration). *A map $i : A \to X$ is a cofibration if given $f : X \to Y$ and $H : A \times I \to Y$ such that $H(a, 0) = f(i(a))$ for all $a \in A$ then $\exists \tilde{H} : X \times I \to Y$ such that $\tilde{H}(i(a), t) \to H(a, t)$ and $H(x, 0) = f(x)$.*

Theorem 1.6. *Let (X, A), $\iota : A \to X$ the inclusion, be a cofibration. Then if A is a weak retract of X, then it is a retract of X.*

Proof. Let $r : X \to A$ satisfy that $r\iota \simeq \mathrm{id}_A$ and let H be that homotopy. We get the following diagram:



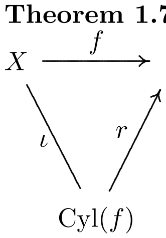
Let $r' = \tilde{H}(x, 1)$, and $r' \simeq r$ and we get $r'\iota = \mathrm{id}_A$. □

We will now do a construction to check the WLOG in Whitehead 2.

$f : X \to Y$ any map. Define $\mathrm{Cyl}(f) \equiv ((X \times I) \coprod Y)/((x, 1) \sim f(x))$.

Then $\iota : X \to \mathrm{Cyl}(f)$ by $x \mapsto (x, 0)$ is an inclusion and $j : Y \to \mathrm{Cyl}(f)$ by $j(y) = y$. Then $r : \mathrm{Cyl}(X) \to Y$, $r(x, t) = f(x)$ and $r(y) = y$.

Theorem 1.7. *There exists a commutative diagram*



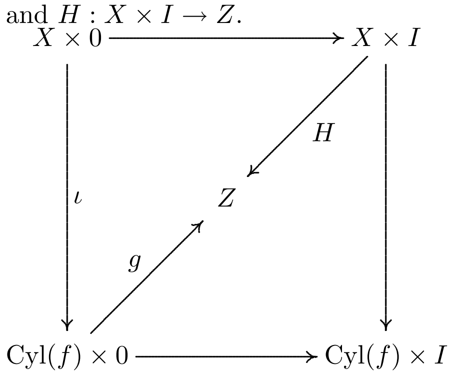
Such that

1. $r \circ \iota = f$
2. $jr \simeq \mathrm{id}_{\mathrm{Cyl}(f)}$ *rel* Y*, and so* $Y \simeq \mathrm{Cyl}(f)$
3. ι *is a cofibration*

Proof. The first part is trivial.

$jr(x,t) = f(x) \in Y$ and $gr(y) = y$, so we set $H((x,t),s) = (x,(1-s)t+s)$ and $H(y,s) = y$.

So all that remains is the cofibration part. Suppose that $g : \mathrm{Cyl}(f) \to Z$ and $H : X \times I \to Z$.



Define $\tilde{H}(y,s) = g(y)$ for $(y,s) \in Y \times I$, $\tilde{H}((x,t),s) = g((x, \frac{2t-s}{2-s}))$ for $0 \le s \le 2t \le 2$ and as $g((x, \frac{s-2t}{1-t}))$ for $0 \le 2t \le s \le 1$. □

2 Lecture 3

Recall: Every cofibration $A \to X$ is an injection.

Up to homotopy, every map is a cofibration.

Proposition 2.1. $A \to X$ *is a cofibration iff the map* $A \times I \cup X \times \{0\} \to X \times I$ *has a retraction.*

Proof. Use the HEP in each direction. □

Lemma 2.2. $A \to A \cup_f e^n$ *is a cofibration. More generally,* $A \to A \cup_f \cup_\alpha e^n_\alpha$ *is a cofibration.*

Proof. $\partial e^n \to e^n$ is a cofibration, since $\partial e^n \to \mathrm{Cyl}(\partial e \setminus pt)$ is a cofibration

By the proposition, there exists a retraction of $e^n \times I \xrightarrow{r} \partial e^n \times I \cup e^n \times \{0\}$.

We need to show that $A \cup_f e^n \times I$ retracts onto $A \times I \cup A \cup_f e^n \times \{0\}$.

$(A \cup_f e^n) \times I \simeq A \times I \cup_{f \times \mathrm{id}_I} (e^n \times I)$. Define $\tilde{r} : A \times I \cup_{f \times \mathrm{id}_I} (e^n \times I) \to A \times I \cup (A \cup_f e^n \times \{0\})$. □

Define a relativie CW complex (X,A) start with A and attach 0-cells to get a 0-skeleton, $(X,A)^0$. $(X,A)^k$ is built from $(X,A)^{k-1}$ by attaching k-cells.

Proposition 2.3. *If (X, A) is a relative CW complex, then $A \to X$ is a cofibration.*

Proof. Is by induction, since a composition of cofibrations is a cofibration. □

Recall that $\pi_k(X, x_0) \equiv [(I^k, \partial I^k), (X, x_0)]$ =homotopy classes of maps.

A homotopy H between $\alpha, \beta : (I^k, \partial I^k) \to (X, x_0)$, $H : (I^k, \partial I^k) \times I \to (X, x_0)$ such that $H(x, 0) = \alpha(x)$ and $H(x, 1) = \beta(x)$.

$\alpha + \beta(x_1, \ldots, x_k) = \alpha(2x_1, \ldots, x_k)$ on $x_1 \in [0, 1/2]$ and $\beta(2x_1 - 1, \ldots, x_k)$ on $[1/2, 1]$.

Lemma 2.4. *If x, y are in the same path component, then $\pi_k(X, x) \simeq \pi_k(X, y)$.*

Proof. If γ is a path from $x \to y$, define a map from $\pi_k(X, x) \to \pi_k(X, y)$ by SEE PICTURE IN HATCHER Check that $[\gamma\alpha]$, the homotopy class of $\gamma\alpha$ only depends on the homotopy class of α, and that $[\gamma, \alpha]$ only depends on $[\gamma$ rel $x, y]$. □

Thus, $\pi_1(X, x)$ acts on $\pi_k(X, x)$ for all k.

Proposition 2.5. *Let $p : (Y, y) \to (X, x)$ be a covering space. Then $p_* : \pi_k(Y, y) \to \pi_k(X, x)$ is an isomorphism for $k \geq 2$.*

Proof. If we have a map $f : Z \to X$ then a lift $\tilde{f} : Z \to Y$ exists when $f_*\pi_1(Z) \subset p_*\pi_1(Y)$.

Let $\alpha : S^k \to X$ represent a homotopy class, then, since $\pi_1(S^k) = 0$, it lifts to $\tilde{\alpha} : S^k \to Y$, so p_* is surjective.

To show injectivity, let $\alpha : S^k \to Y$ be such that $p\alpha \simeq c_x$ the constant map. Let H be a homotopy between $p\alpha$ and c_*.

By the homotopy lifting property, we get $\tilde{H} : S^k \times I \to Y$, so $\alpha \simeq c_y$. □

Proposition 2.6. *If $\{X_\alpha, x_\alpha\}$ is a collection of pointed spaces, then*

$$\pi_k(\prod_\alpha X_\alpha, \{x_\alpha\}) \simeq \prod_\alpha \pi_k(X_\alpha, x_\alpha)$$

.

Proof. This is true by the definition of product, looking at the space of maps. □

Definition 2.1 (Aspherical). *A space X such that $\pi_k(X) = 0$ for $k \geq 2$ is called aspherical*

Proposition 2.7. *If X and Y are two aspherical CW complexes and $\pi_1(X) \simeq \pi_1(Y)$, then $X \simeq Y$.*

Conjecture 2.1 (Borel). *If X^n, Y^n are two aspherical manifolds such that $\pi_1(X) \simeq \pi_1(Y)$, then X is homeomorphic to Y.*

How to construct aspherical manifolds:

Start with a contractible universal cover M, say, $\mathbb{R}^n$. Find a proper, free action of a group Γ on M and form the quotient M/Γ.

For instance, $M = Sl_2(\mathbb{R})/SO(2) \simeq \mathbb{H}^2$ and $\Gamma = PSL_2(\mathbb{Z})$, and then $\mathbb{H}^2/\Gamma$ is aspherical.

Theorem 2.8 (Cartan)**.** *If X is a compact Riemannain manifold such that $\kappa \leq 0$ (sectional curvature), then X is aspherical.*

Let (X, A, x_0) be a pointed pair. Then $\pi_k(X, A, x_0)$ is the set of homotopy classes of maps $(I^k, \partial I^k, \Lambda^{k-1}) \to (X, A, x_0)$ where $\Lambda^{k-1} = I \times \partial I^{k-1} \cup \{0\} \times I^{k-1}$.

This is a set for $k = 1$, a group for $k = 2$ and an abelian group for $k \geq 3$.

Theorem 2.9. *There exists a long exact sequence*

$$\to \pi_{k+1}(X, A, x_0) \to \pi_k(A, x_0) \to \pi_k(X, x_0) \to \pi_k(X, A, x_0) \to \dots$$

$$\to \pi_2(X, x_0) \to \pi_2(X, A, x_0) \to \pi_1(A, x_0) \to \pi_1(X, x_0)$$

.

3 Lecture 4

We will be talking about Function Spaces.

If X, Y are topological spaces, then we can look at $Map(X, Y)$ the set of all continuous maps from X to Y. You can topologize this by giving it the "compact-open." It has a subbasis of sets of the form for each $\subset X$ compact and $O \subset Y$ open, the set $U_{K,O} = \{f : X \to Y | f(K) \subset O\}$. Take the smallest topology containing all of these.

Facts:

1. If Y is metric, then the compact-open topology coincides with the metric topology $d(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$.
2. We want the following to be true: $Map(X \times Y, Z) \simeq Map(X, Map(Y, Z))$. We think of this as being analagous to the fact that $\hom_R(E \otimes_R F, G) \simeq \hom_R(E, \hom_R(F, G))$, that is, that $- \times Y$ is left adjoint to $Map(Y, -)$. This adjoint property is correct if you work in the category of compactly generated spaces.

Examples:

1. $Map(X \times I, Y) \simeq Map(X, Map(I, Y))$, that is, the space of maps from $X \times I$ to Y is homeomorphic to the space of maps from X into the maps from I to Y.

2. If X, Y are pointed sets, then $Map_0(X, Y)$ are pointed maps, then we have $Map_0(X \wedge Y, Z) \simeq Map_0(X, Map(Y, Z))$, where $\wedge$ is the smash product. The smash product is $X \wedge Y = X \times Y/(X \times \{y_0\} \cup \{x_0\} \times Y)$. So $S^1 \wedge X = S^1 \times X/\{*\} \times X \cup S^1 \times \{x_0\} \simeq \sum X$, the suspension of X, which is $X \times I/(X \times \{0\} \simeq p_0, X \times \{1\} \simeq p_1)$. We call $S^1 \wedge X$ the reduced suspension.

   Lemma 3.1. *If $* \to X$ is a cofibration, then $\sum X \simeq S^1 \wedge X$.*

3. If X is a space, then $Map_0(S^1, X) \equiv \Omega X$.

   Lemma 3.2. $\pi_n(X) \simeq \pi_{n-1}(\Omega X)$

   Proof. $[(S^n, *), (X, x_0)] = Map_0(S^n, X)$/homotopy is the same as the set $Map_0(S^1 \wedge S^{n-1}, X)$/homotopy which is homeomorphic to the set $Map_0(S^{n-1}, Map_0(S^1, X))$/homotopy$= \pi_{n-1}(\Omega X)$. □

   Corollary 3.3. $\pi_n(X) = \pi_{n-1}(\Omega X) = \ldots = \pi_0(\Omega^n X)$.

ΩX is a group up to homotopy, ie, an H space.

Definition 3.1 (H-space)**.** *(Y, m) is an H-space if $m : Y \times Y \to Y$ is a continuous map which satisfies all of the categorical group axioms up to homotopy. (sometimes without inverses)*

Set $PX = Map_0(I, X)$ and call $Map(S^1, X)$ the free loop space.

Last time we defined $\pi_n(X, A, x_0) = [(I^n, \partial I^n, J^{n-1}), (X, A, *)]$, and stated the existence of a long exact sequence.

Lemma 3.4. *$f : (D^n, S^{n-1}, *) \to (X, A, x_0)$ represents 0 in $\pi_n(X, A, x_0)$ iff f is homotopic rel S^{n-1} to a map whose image lies in A.*

Proof. $\Leftarrow$: $f \simeq g$ rel S^{n-1}, $[f] = [g]$ and $g \simeq c_{x_0}$ by composing g with the deformation retraction of D^n onto $* \in S^{n-1}$.

$\Rightarrow$: Suppose that $f : (D^n, S^{n-1}, *) \to (X, A, x_0)$ represents 0 in $\pi_n(X, A, x_0)$, then $f \simeq c_{x_0}$ rel S^{n-1}, that is, there exists a homotopy $H : (D^n \times I, S^{n-1} \times I, * \times I) \to (X, A, *)$ such that $H(z, 0) = f$ and $H(z, 1) = c_{x_0}$. So that $f \simeq H|_{D^n \times \{1\} \cup S^{n-1} \times I}$. □

Definition 3.2 (n-connected)**.** *A space (X, x_0) is called n-connected if we have that $\pi_k(X, x_0) = 0$ for all $k = 0, \ldots, n$. To be 0-connected means path connects, and to be 1-connected is simply connected.*

We say that (X, A) is n-connected if $\pi_k(X, A, x_0) = 0$ for $k = 1, \ldots, 0$ and that it is 0-connected, where it is 0-connected if each path compoentnt of X intersects A.

Proposition 3.5. *Let (X, A) be a relative CW pair, (Y, B) any pair with $B \neq \emptyset$. Assume that for each k such that (X, A) has k-called that $\pi_k(Y, B) = 0$. Then any map $f : (X, A) \to (Y, B)$ is homotopic rel A to a map where the image lies in B.*

Proof. Start with an induction argument. If (X, A) is 0-connected, set $(X, A)^0 = A \cup$ 0-cells. So $f|_{(X,A)^0} \simeq$ to a map whose image lies in A.

Assume that $f : (X, A) \to (Y, B)$ has been homotoped rel $(X, A)^{n-1}$ to a map f_{n-1} with $f_{n-1}((X, A)^{n-1}) \subseteq B$.

We want to see that this homotopes further to a map f_n with $f_n((X, A)^n) \subset B$.

Assume that e^n is an n-cell of $(X, A)^n$. Then $(e^n, \partial e^n) \xrightarrow{i} (X, A)^n \xrightarrow{f_{n-1}} (Y, B)$ the image $[f_{n-1}, i] \in \pi_n(Y, B) = 0$, and this implies that $f_{n-1}|_{(X,A)^{n-1} \cup e^n}$ can be homotoped rel ∂e^n to a map whose image is in B.

We can do this for all cells of dimension n, and so we get a map $f_n : (X, A)^n \to B$ and a homotopy from f_{n-1} to f_n rel $(X, A)^{n-1}$, and so by HEP, as $(X, A)^{n-1} \to (X, A)^n$ is a cofibration. This homotopy extends to all of X.

We have a seuqence of maps f_n and homotopies H_n from f_n to f_{n+1}. Define on $X \times I$ a homotopy $H(x, t)$ which is the n^{th} homotopy for $t \in [1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}]$.

So on any finite skeleton $(X, A)^n$ the homotopy becomes stable after the n^{th} interval, so this extends to a homotopy on $X \times I$. □

Proposition 3.6. $\pi_k(S^n, *) = 0$ *for all* $k < n$.

Proof. If $f : N \to M$ is a map of compact differentiable manifolds, then it is homotopic to a differentiable map by Stone-Weierstrass. S othere exists a differentiable map $g : N \to U \subset M$ which is as close as you want to f. So now we have a homotopy between f and g. If g is a differentiable map, then by Sard's Theorem, it misses a point, and so if we retract $S^n \setminus \{*\}$ to a point, we homotope g to a point, and so f is homotopic to a constnat map. □

4 Lecture 5

Theorem 4.1. *Let* (X, A) *be a relative CW-pair and* (Y, B) *an arbitrary pair with* $B \neq \emptyset$. *Suppose that* $\pi_k(Y, B) = 0$ *in all dimensions such that* (X, A) *has a* k*-cell. Then any map* $f : (X, A) \to (Y, B)$ *is homotopic rel* A *to a map* $g : X \to B$.

Theorem 4.2 (Whitehead)**.** *If* $f : X \to Y$ *is a map between CW-complexes which induces an isomorphism on homotopy* $f_* : \pi_k(X, x_0) \simeq \pi_k(Y, y_0)$ *for all* k*, then* f *is a homotopy equivalence. Moreover, if* $f : X \to Y$ *is an inclusion of a subcomplex, then* Y *deformation retracts to* X.

Proof. First assume that $f : X \to Y$ is such an inclusion. Then consider $(Y, X) \to (Y, X)$. $\pi_k(Y, X) = 0$ by the long exact sequence on homotopy, and then we apply the theorem to the inclusion, getting that it is homotopic to g rel X such that $g : Y \to X$ is a deformation retract.

Now assume that $f : X \to Y$ is cellular. Then Cyl(f) gives $X \to Cyl(f) \simeq Y$ is a subcomplex, and so the result follows from the previous result.

The general result follows from the Cellular Approximation Theorem. □

Theorem 4.3 (Cellular Approximation Theorem)**.** *$f : X \to Y$ is any map between CW complexes. Then it is homotopic to a cellular map.*

Lemma 4.4. *Let $f : e^n \to Z = W \cup e^k$ for $n < k$. Then $f$is homotopic rel $f^{-1}(W)$ to a map $f_1 : e^n \to Z$ for which there exists a ball $B \subset e^n$ such that*

1. *$f_1(B) \subset e^k$ and $f_1|_B$ is differentiable*
2. *$B \supset f_1^{-1}(U)$ for some U a nonempty open subset of e^n.*
3. *f_1_B misses some point of U.*

We can now prove the Cellular Approximation Theorem.

We assume that f is cellular on X^{n-1}. Let e^n be an n-cell of X with e^n compact, $f(e^n)$ is compact, and so finitely many cells in Y meet $f(e^n)$ closure. We don't have to worry about cells of dimension $\leq n$. Let e^k be the alrgest dimensional cell which meets $f(e^n)$ and $k > n$.

Then $f : X^{n-1} \cup e^n \to Y$ is homotopic rel X^{n-1} to a map f_1 which misses a point of e^k, call that point p.

So there exists a deformation of $e^k \setminus p$ to $Y \setminus e^k$, and this gives a homotopy of f on $X^{n-1} \cup e^n$ to a map on $Y \setminus e^k$, and we can replace f by this.

Do this for all the other cells that $f(e^n)$ hits. So we arrive at a map $f : X^{n-1} \cup e^n \to Y^n$. Now do this for all n-cells. So we get that our original f is homotopic to a map which is cellular on X^n. Do this for each dimension.

This also works for relative pairs.

For a while, we will write the homotopy extension property as: $p : Y \to X$ has the HLP, if you have the following diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow p \\ A \times I & \xrightarrow{H} & X \end{array}$$

Definition 4.1 (Fibration)**.** *A map $p : Y \to X$ is called a fibration if it has the HLP with respect to all spaces A.*

Fact: $p : X \times F \to X$ is always a fibration.

If (A, B) is a pair, then p has HLP for (X, A) if

$$\begin{array}{ccc} A \times \{0\} \cup B \times I & \xrightarrow{f} & Y \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow p \\ A \times I & \xrightarrow{H} & X \end{array}$$

Proposition 4.5. *$p : Y \to X$ is a Serre fibration if it has the relative HLP for all pairs $(D^n, \partial D^n)$, which is iff HLP for all CW complexes A.*

Lemma 4.6. *$p : Y \to X$ has the relative HLP with respect to $(D^n, \partial D^n)$ iff HLP with respect to D^n.*

Proof. $(D^n \times I, D^n \times \{0\} \cup \partial D^n \times I) \simeq (D^n \times I, D^n)$. □

Definition 4.2 (Fiber Bundle)**.** *$p : Y \to X$ is a fiber bundle with fiber F, if $\forall x \in X$ there exists a neighborhood U of x and a homeomorphism $\varphi_U : p^{-1}(U) \to U \times F$.*

Theorem 4.7. *Any fiber bundle is a Serre fibration.*

Proof. If $p : X \times F \to X$, we just proved it.
... □

Suppose that $A \subset X$. Define $\nu = \{f : I \to X | f(0) \in A, f(t) \notin A \text{ for } t > 0\}$, then we get a fibration $\nu \to A$ by $f \mapsto f(0)$. This is called the homotopy normal bundle.

If $N \to M$ is an embedding of differentiable manifolds, $p : \nu \to N$ and $\pi : S\nu(N \to M) \to N$ (the normal sphere bundle), that is, $\pi^{-1}(x) = S^{m-n-1}$. Then, ν is fiber homotopy equivalent to $S\nu(N \to M)$

Which homotopy types are the homotopy types of compact orientable manifolds?

The big homotopy consequence of manifoldness is Poincare Duality. That is, there exists a fundamental class $[M] \in H_n(M, \mathbb{Z})$ such that $- \cap [M] : H^k(M, \mathbb{Z}) \to H_{n-k}(M, \mathbb{Z})$ is an isomorphism.

Definition 4.3 (Poincare Duality Space)**.** *A finite CW complex is an n dimensional Poincare Duality space iff there exists a class $c \in H_n(X, \mathbb{Z})$ such that $- \cap c : H^k(X, \mathbb{Z}) \to H_{n-k}(X, \mathbb{Z})$ is an isomorphism.*

Let X be a finite CW complex. Embed $X \to \mathbb{R}^N$ for some big N.

Now take $\nu(X \to \mathbb{R}^N) \to X$

Theorem 4.8 (Spivak)**.** *X is a PD space iff $p^{-1}(x)$ is homotopic to a sphere.*

Theorem 4.9 (Browder)**.** *If X is a simply connected $n \geq 5$, then X is homotopic to a manifold iff $h\nu(i)$ is fiber homotopy equivalent to a sphere bundle on X.*

5 Lecture 6

Definition 5.1 (Hurewicz Fibration)**.** *p is a Hurewicz Fibration if it has teh HLP with respect to all spaces.*

Theorem 5.1 (Def of Serre Fibration)**.** *The Following are equivalent*

1. *$p : Y \to X$ is a Serre Fibration, that is, it has HLP with respect to discs.*
2. *$p : Y \to X$ has the HLP wrt to all CW complexes*

3. $p : Y \to X$ *has the relative HLP wrt all pairs* $(D^n, \partial D^n)$

4. $p : Y \to X$ *has relative HLP wrt* (Z, A) *where* $A \subset Z$ *is a subcomplex of a CW complex* Z *where the inclusion is a homotopy equivalence.*

From now on, a fibration will always be a Serre fibration.
Construction: Let $f : X \to Y$ be any map.
Facts: If $p : E \to Y$ is a fibration, then $f^*E \to X$, is also a fibration.

Definition 5.2 (Homotopy Fiber)**.** *If* $f : X \to Y$ *is any map, the homotopy fiber of* f *is the fiber of* $p : X' \to Y$ *where* $X' \simeq X$ *and* p *is a fibration. This is a well-defined homotopy type.*

Theorem 5.2. *Given a fibration* $p : Y \to X$ *with* $y_0 \mapsto x_0$*, there exists a long exact sequence* $\pi_k(F, y_0) \to \pi_k(Y, y_0) \to \pi_k(X, x_0) \to \dots$*.*

Proof. Look at the map $p : (Y, F, y_0) \to (X, x_0, x_0)$. We claim that it is an isomorphism on homotopy groups.

It is surjective: Let $f : (I^k, \partial I^k) \to (X, x_0)$. Apply the relative HLP. There exists $\tilde{f}$ with $p\tilde{f} = f$ and $p\tilde{f}(\partial I^k) = x_0$, so $\tilde{f} : \partial I^k \to F$, and so $\tilde{f} \in \pi_k(Y, F, y_0)$ which lifts f.

It is injective: Assume we have $\tilde{f}_0$ and $\tilde{f}_1$ with $[p\tilde{f}_0] = [p\tilde{f}_1]$ in $\pi_k(X, x_0)$. Let $H : (I^k, \partial I^k) \times I \to X$ be the homotopy taking $p\tilde{f}_0$ to $p\tilde{f}_1$. So we want to lift the homotopy to Y. We use the relative HLP. □

Example 5.1. $S^1 \to S^3 \to S^2$ *is the Hopf Fibration, and so we get that* $\pi_{k+1}(S^2) \to \pi_k(S^1) \to \pi_k(S^3) \to \pi_k(S^2) \to \dots$*. As* $\pi_j(S^1) = 0$ *for* $j > 1$*, we have that for* $k > 2$*,* $\pi_k(S^3) \to \pi_k(S^2)$ *is an isomorphism.*

So this says that $\pi_3(S^2) = \mathbb{Z}$

Unfortunately, excision fails, so homotopy is harder here than homology.

Another good source of examples are quotients of a manifold by a compact Lie group.

Suppose that G acts freely and differentiably on X, then $G \to X \to G/X$ is a fiber bundle with fiber G. If $H \subset G$ is a closed subgroup, of a compact lie group, then $H \to G \to G/H$ is a fiber bundle.

We will eventually prove the Hurewicz Theorem via the Serre Spectral Sequence.

Theorem 5.3 (CW Approximation Theorem)**.** *Given any space* X*, there exists a CW complex* Z *and a map* $f : Z \to X$ *which is a weak homotopy equivalence.*

Definition 5.3. f *is a weak homotopy equivalence if* $\pi_k(Z, z) \to \pi_k(X, f(z))$ *is an isomorphism for all* k *and all* z*.* X *and* Y *are weak homotopy equivalent if there exists a weak homotopy equivalence between them.*

Moreover, weak homotopy equivalence is an equivalence relation.

6 Lecture 7

Definition 6.1 (CW Model)**.** *Let (X, A) be a pair with A a CW complex. We say that (Z, A, f) is an n-connected CW model for (X, A) if*

1. *Z is a CW complex*
2. *(Z, A) is n-connected*
3. *$f : Z \to X$ such that $f|_A = \mathrm{id}_A$ and $f_* : \pi_k(Z, z) \to \pi_k(X, x)$ is injective for $k \geq n$ and surjective for $k > n$.*

Since (Z, A) is n-connected we have injective for $k < n$ and surjective for $k \leq n$.

Theorem 6.1. *Every (X, A) for $A \neq \emptyset$ has an n-connected CW model for each n.*

Proof. We will construct a sequence of CW complexes $Z_n \subset Z_{n+1}$ with $f_n : Z_n \to X$ such that Z_{k+1} is obtained from Z_k by attaching e^{k+1}'s and $f_k : \pi_i(Z_k) \to \pi_i(X)$ is injective for $i < k$ and surjective for $n < i \leq k$.

Set $Z_n = A$ and $f_n = \mathrm{id}_A$. By induction, suppose that we've constructed Z_k satisfying the pushforward condition. We will now construct Z_{k+1}. $f_k : Z_k \to X$ satisfies the pushforward conditions.

$f_k : \pi_k(Z_k) \to \pi_k(X)$ is not necessarily an injection. Let $\varphi_\alpha : S^k \to Z_k$ be a collection of maps such that $[\varphi_\alpha]$ generates $\ker \pi_k(Z_k) \to \pi_k(X)$. Let $Y_{k+1} = Z_k \cup \coprod e_\alpha^{k+1}$ glued by φ_α.

$f_k\varphi_\alpha : S^k \to X$ are all homotopically trivial, and so f_k extends to $Y_{k+1} \to X$. Now the map $\pi_k(Y_{k+1}) \to \pi_k(X)$ is injective, and its still surjective. We still don't have $f_{j+1} : Y_{k+1} \to X$ is surjection on π_{k+1}.

So choose generators $\psi_\beta : S^{k+1} \to X$ for $\pi_{k+1}(X)$. Let $Z_{k+1} = Y_{k+1} \vee_\beta S_\beta^{k+1}$. f_{k+1} extends to Z_{k+1} by setting it equal to ψ_β on S_β^{k+1}. These archieve the surjectiveity, and set $Z = \cup Z_n$. □

Corollary 6.2. *Any X has an $f : Z \to X$ where Z is CW and f is an isomorphism on all homotopy groups.*

Proof. Set $A = pt$ and $n = 0$. □

Definition 6.2 (Eilenberg-MacLane Space)**.** *Let G be a group. A space $K = K(G, n)$ is an Eilenberg-MacLane space if $\pi_k(K) = G$ for $k = n$ and 0 else.*

$K(\tilde{G}, 1)$ is contractible. G acts on this space freely and properly. So to construct a $K(G, 1)$, we need to find a contractible space on which G acts freely and properly.

$K(\mathbb{Z}, 1) = \mathbb{R}/\mathbb{Z} = S^1$, $K(\mathbb{Z}^n, 1)$ is the n-torus, $K(\mathbb{Z}/2, 1) = \mathbb{RP}^\infty$.

Look at the generalized Hopf fibration $S^1 \to S^{2n+1} \to \mathbb{CP}^n$. The LES of a fibration gives $\pi_i(S^1) \to \pi_i(S^{2n+1}) \to \pi_i(\mathbb{CP}^n) \to \pi_{i-1}(S^1)$.

Taking these in increasing n, we get $\pi_i(\mathbb{CP}^\infty) = 0$ for $i > 2$. And also, $\pi_1(\mathbb{CP}^\infty) = 0$ and $\pi_1(\mathbb{CP}^\infty) = \mathbb{Z}$ from the LES. So $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$.

We will see that $\check{H}^n(X, G) \cong [X, K(G, n)]$ as groups.

Why is this a group? (on the right)

Theorem 6.3. *For any group G, $K(G, 1)$ exists. For any abelian group G, $K(G, n)$ exists for any n.*

Proof. Exactly the same as the proof of the cellular approximation tehorem. Take generators for G, g_α, to form $Z_n = \vee S^n_\alpha$, etc □

Let $X = K(G, n)$. Look at the fibration $\Omega X \to PX \to X$. Then we have $\pi_{k+1}(PX) \to \pi_{k+1}(X) \to \pi_k(\Omega X) \to \pi_k(PX)$, so we have $\pi_{k+1}(X) \simeq \pi_k(\Omega X)$, and thus ΩX is a $K(G, n-1)$.

So then above, $[X, K(G, n)] = [X, \Omega K(G, n+1)] = [X, \Omega\Omega K(G, n+2)] = [X, \Omega^k(n+k)]$.

Let H be an infinite dimensional complex Hilbert Space. Then $S(H)$ is the Sphere in H, and it is contractible. $H = L^2([0, 1])$, and $\xi_0(x) = 1$, $\xi_0 \in S(H)$. Look at $U(1) \to S(H) \to \mathbb{CP}(H)$.

Definition 6.3 (Double Complex)**.** *A double complex is a collection of objects $C^{p,q}$ such that the "rows" (with q fixed) are complexes with differential d and the columnes (with p fixed) are complexes with differential δ such that $d\delta + \delta d = 0$, and $p, q \geq 0$.*

Define $\mathrm{Tot}^n(C) = \oplus_{p+q=n} C^{p,q}$ and $D = d + \delta$ a map from $\mathrm{Tot}^n \to \mathrm{Tot}^{n+1}$, the total complex. $H^*(\mathrm{Tot}\, C) = H^(C)$.

Let C be a double complex. Add obkects A^i to the ith row such that the rows are exact and $A^i \to C^{0,i}$ is an inclusion. Assume that the rows are exact. Then $H^*(A) \simeq H^*(\mathrm{Tot}\, C)$.

Proof by diagram chase.

7 Lecture 8

Next time, Jason Devito will speak about Framed Cobordism and homotopy groups.

Definition 7.1 (Cobordism)**.** *Two compact manifolds M_1, M_2 are called cobordant if thereexists a compact manifold W with $\partial W = M_1 \coprod M_2$.*

If M_1 and M_2 are oriented, then want W oriented and $\partial W = M_1 \coprod \bar{M}_2$.

Cobordism is an equivalence relation, and define N^k = the group of dimension k cobordism classes of manifolds.

$[M] + [N] = [M \coprod N]$, and $[M] + [M] = [M \coprod M] = [\partial M \times I] = 0$.

Proposition 7.1. *$M \coprod N$ is cobordant to a connected manifold.*

Proof. In fact, claim that $M \coprod N$ is cobordant to $M \# N$. □

Now back to Spectral Sequences...

Proposition 7.2. *Let C^{pq} be a double complex with δ the horizontal and d the vertical differentials. Let C^i be a complex to the left of it mapping in horizontally (an augmented complex) such that we can augment again by 0 to make the rows exact, then $\epsilon : (C, d) \to \mathrm{Tor}(C, D)$ induces an isomorphism on cohomology.*

Proof. Let $C \in \mathrm{Tor}^n(C, D)$, then $c = c^n + c^{n-1} + \ldots + c^0$ and $Dc = 0$. Diagram chase. □

Varations: Other quadrants.

If we have a third and fourth quadrant complex, how do we form Tot? If we want this lemma, we take product rather than sum.

From Spectral Sequences, DeRham's Theorem is easy:

First, we do Cech Cohomology.

Definition 7.2 (Cech Cohomology)**.** *Let X be a space and $\mathcal{U} = \{U_\alpha\}$ a cover of X. Define $\check{C}^q(\mathcal{U}) = \{f(\alpha_{i_0}, \ldots, \alpha_{i_q}) \in R$ with f antisymmetric and equal to zero if $U_{i_0} \cap \ldots \cap U_{i_q} = \emptyset$. This gives a complex. We then take $\check{H}^q(X, R) = \varinjlim H^r(\check{C}(\mathcal{U}, R))$, where the limit is over all covers.*

We will prove the deRham Theorem:

Let M be a manifold and suppose that $\mathcal{U}$ is a locally finity cover.

Let λ_α be a partition of unity subordinate to $\mathcal{U}$. Then $0 \to \Omega^q(M) \xrightarrow{\epsilon} \prod_{\alpha_0} \Omega^q(U_{\alpha_0}) \to \mathrm{ord}_{\alpha_0<\alpha_1} \Omega^q(U_{\alpha_0} \cap U_{\alpha_1}) \to \ldots$.

Define $(\delta\omega)_{\alpha_0,\ldots,\alpha_{\ell+1}} = \sum_{i=1}^{\ell+1} (-1)^i \omega_{\alpha_0,\ldots,\hat{\alpha}_i,\ldots\alpha_{\ell+1}}|_{U_{\alpha_0,\ldots,\alpha_{\ell+1}}}$

Also define $\epsilon(\omega)_{\alpha_0} = \omega|_{U_{\alpha_0}}$. Note that $\delta^2 = 0$.

Lemma 7.3 (Poincare Lemma)**.** *Let $U \subset M$ be diffeomorphic to a ball in $\mathbb{R}^n$. THen $H^i(\Omega^*(U), d) \simeq \mathbb{R}$ if $i = 0$ and 0 otherwise.*

Let M be a manifold, cover M by U_{α_i} such that their intersections are all diffeomorphic to balls in $\mathbb{R}^n$. Can't do this for all manifolds (ie, Long Line), but can for compact ones.

Take the double complex $C^{pq} = \prod_{\alpha_0,\ldots,\alpha_{p-1}} \Omega^{q-1}(U_{\alpha_0,\ldots,\alpha_{p-1}})$ with vertical maps d and horizontal maps $(-1)^p\delta$.

We can augment this complex by $\Omega^i(M)$, and so $H^*(\Omega^* M, d) \simeq H^*(\mathrm{Tot}\,\check{C}\Omega)$ and we can augment below by p copies of $\mathbb{R}$, and these are exact by Poincare, and so we have the deRham Theorem.

Ext

Let R be a ring and M, N two left modules, then $\mathrm{Ext}_R(M, N)$ is defined by taking a projective resolution of M, taking $\hom(-, N)$ with this resolution, and then taking cohomology. H^i of this complex is $\mathrm{Ext}^i_R(M, N)$.

We can also define it with an injective resolution of N, in precisely the same way.

Theorem 7.4. $\mathrm{Ext}^*_R(M, N)$ *is independent of what projective or injective resolution you take or which type you take.*

Proof. Let $P^* \to M$ and $N \to I^*$ be projective and injective resolutions. Double complex $C^{pq} = \hom(P^{p-1}, I^{q-1})$, and augment on left by $\hom(P^i, N)$ and below by $\hom(M, I^j)$. □

Let G be a group, M a G-module and $C^i(G, M) = \{\varphi : G^i \to M\}$ as in Cech Cohomology.

Theorem 7.5. *Let G be a group, R a ring trivial as a G-module. Let X be a CW-complex which is a $K(G, 1)$, then $H(G, R) \simeq H(X, R)$*

8 Lecture 9

Our goal today is a theorem of Pontryagin:

Theorem 8.1 (Pontrjagin)**.** *M^{n+k} compact with no boundary (closed). There is a one to one correspondence between $[M^{n+k}, S^k]$ and framed compact submanifolds $N^n \subset M^{n+k}$ modulo framed cobordism.*

Definition 8.1 (Cobordism)**.** *The manifolds $N^n, (N')^n \subset M^{n+k}$ are cobordant if $N \times [0, \epsilon)$ and $N' \times (1 - \epsilon, 1]$ can be extended to a manifold $X \subset M \times [0, 1]$ such that $\partial X = N \times \{0\} \cup N' \times \{1\}$.*

Definition 8.2 (Framing)**.** *A framing of $N \subset M$ is a trivialization of the normal bundle of N in M.*

Notationally, (N, v) where $v = (v^1(x), \ldots, v^k(x))$ for $x \in N$, v is a basis of $(T_xN)^\perp$.

So a framed cobordism is a cobordism X with a framing w with $w|_{\partial X}$ is v or v'.

Notationally, we will almost always omit the framing as understood.

To define $\varphi : [M^{n+k}, S^k] \to$framed submanifolds modulo framed cobordism we take $[f]$ and choose a representative f. We choose a regular value $y \in S^k$ of f, then $f^{-1}(y)$ is a compact n-manifold called the Pontrjagin Manifold of f.

Choose an oriented basis $w^1, \ldots, w^k$ of T_yS^k. Then for $x \in f^{-1}(y)$, $(d_xf) \cdot T_xf^{-1}(y) \oplus (T_xf^{-1}(y))^\perp \to T_yS^k$, then the restriction $d_xf|_{T_xf^{-1}(y)} \equiv 0$.

So $d_xf : (T_xf^{-1}(y))^\perp \xrightarrow{\cong} T_yS^k$. Let $v^i = (d_xf)|^{-1}_{(T_xf^{-1}(y))^\perp}(w^i)$.

So we must check that φ is well defined. That it is independent of choice of basis, for z a regular value near y, we should have $f^{-1}(y) \sim f^{-1}(z)$, if $f \simeq g$ and y is a common regular value, then $f^{-1}(y) \sim g^{-1}(y)$, and y, z regular implies that $f^{-1}(y) \sim f^{-1}(z)$.

For the first one, if we let w and u be two choices of positively oriented bases T_yS^k, then $w, u \in GL^+(k)$ where $GL^+(k)$ is connected, and so path connected. Let $\gamma : [0, 1] \to GL^+(k)$ by w for $[0, \epsilon)$ and u for $(1 - \epsilon, 1]$. Then $(f^{-1}(y), f^*w) \sim (f^{-1}(y), f^*u)$ via $X = (f^{-1}(y) \times [0, 1], f^*\gamma(t))$.

For the second, we must first define "near". By Sard's Theorem, there exists $U \subset S^k$, $y \in U$, such that $\forall x \in U$, x is regular.

WLOG, let $U = \{x | \|y - x\| < \epsilon\}$ and assume that $z \in U$. Choose a one parameter family of rotations r_t such that $r_1(y) = z$ and $r_t = \text{id}$ for $t \in [0, \epsilon)$, and $r_t = r_1$ for $t \in (1 - \epsilon, 1]$. Then $r_t^{-1}(z)$ lies on the great circle from z to y.

Let $F : M \times [0,1] \to S^k$ by $F(x,t) = r_t \circ f(x)$, as z is a regular value of f, it is for F. Thus $(F^{-1}(z), F^*v)$ is a framed cobordism between $f^{-1}(z)$ and $f^{-1}(y)$.

Assume that $f \simeq g$ and y is regular for both. Choose a homotopy $F(x,t)$ which is $f(x)$ for $t \in [0, \epsilon)$ and $g(x)$ for $(1 - \epsilon, 1]$. By Sard's Theorem, there eis a z regular for F near y. We know that $f^{-1}(y) \sim f^{-1}(z)$ and $g^{-1}(y) \sim g^{-1}(z)$ and $F^{-1}(z)$ is a framed cobordism between $f^{-1}(z)$ and $g^{-1}(z)$.

Let y, z be regular values of f and let r_t be the rotation $r_1(y) = z$, with $r_t = \text{id}$ for $t \in [0, \epsilon)$ and $r_t = r_1$ for $(1 - \epsilon, 1]$. Then $F(t,x) = r_t \circ f(x)$. And so $f^{-1}(z) = F^{-1}(z) \sim F^{-1}(z) = (r_1 \circ f)^{-1}(z) = f^{-1} \circ r_1^{-1}(z) = f^{-1}(y)$.

We will need the following to proceed:

Lemma 8.2 (Product Neighborhood Theorem)**.** *If $N^n \subset M^{n+k}$ and N is framed, then there exists $V \subset M$ open with $N \subset V$ such that $V \cong N \times \mathbb{R}^k$ such that $h(n, v) = n$.*

SHOWING BIJECTION

9 Lecture 10

Spectral Sequences:

References: Hatcher has notes towards a book on Spectral Sequences on his website.

Book by McCleary "User's Guide to Spectral Sequences"

Definition 9.1 (Spectral Sequence)**.** *A spectral sequence is a sequence of complexes $(E_1, d_1), (E_2, d_2), \ldots$ such that $E_n = H^*(E_{n-1}, d_{n-1})$.*

A morphism of spectral sequences $f_n : (E_n, d_n) \to (F_n, \partial_n)$ has $f_{n+1} = H(f_n)$.

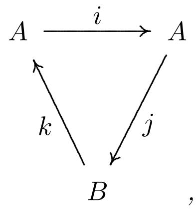
Basic way to construct spectral sequences:

Definition 9.2 (Exact Couple)**.** *An exact couple is (A, B) two groups with $i : A \to A$, $j : A \to B$ and $k : B \to A$ where $\ker j = \operatorname{Im} i$.*

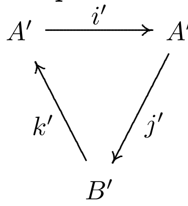
Define $A' = i(A)$. On B, dfefine $d = jk$, then $d^2 = jkjk = 0$. Set $B' = H^*(B, d) = \ker f / \operatorname{Im} d$.

Then we have $A' \xrightarrow{i'} A' \xrightarrow{j'} B$ and so we define $i'(i(a)) = i(i(a))$ and $j'(i(a)) = j(a)$ are well defined. Thus $j(a) = j(a') + jkb$.

Define $k : B' \to A'$ by $k'([b]) = kb$ for $b \in B$ with $db = 0$. So now $db = 0$ implies that $jkb = 0$ which means $kb \in \ker j$, and so $kb \in \operatorname{Im} i \in A'$. And if $b = db' = jkb'$ then $kb = kjk'b' = 0$.



Proposition 9.1. *Given an exact couple* , *it's derived couple*



is exact.

The proof is a diagram chase.

So given an exact couple (A, B), set (A_1, B_1) to be the derived couple, and continue. So (B_n, d_n) form a spectral sequence.

Let $X_0 \subset X_1 \subset \ldots$ be a filtration of a space X. For any p we have $\to H^k(X_p, X_{p-1}) \to H^k(X_p) \to H^k(X_{p-1}) \to H^{k+1}(X_p, X_{p-1}) \to \ldots$. Arrange this in the patter with columns $H^k(X_p)$ with p increasing down the column and k constant and rows $H^{k-2}(X_{p-1}) \to H^{k-1}(X_p, X_{p-1}) \to H^{k-1}(X_p)$ and continuing.

Let (C^*, d) be a cochain complex. A filtration on (C, d) is $C^* = F_0 \supset F_1 \supset F_2 \subset \ldots$ where (F_i, d) are subcomplexes. We set $F_{-k} = C^*$. So now we set $A = \left(\oplus_{p=-\infty}^{\infty} F_p, d\right)$.

So $0 \to F_{p1} \to F_p \to F_p/F_{p+1} \to 0$ is an exact seuqence of complexes. Thus there exists a long exact sequence $H^k(F_{p+1}) \to H^k(F_p) \to H^k(F_p/F_{p+1}) \to \ldots$.

We define $i : A \to A$ to be the sheaf operator. So we have $0 \to A \to A \to \operatorname{coker} i \to 0$. We start with an exact complex $A_0 = H^*(A)$, $B_0 = H^*(\operatorname{coker} i)$ and we have a map $i_0 : A_0 \to A_0$ given by $i_0 = H^*(i)$ and a map $j_0 : A_0 \to B_0$ given by $j_0 = H^*(j)$. So then we get an exact couple using $\delta : H^k(\operatorname{coker} i) \to H^{k+1}(A)$.

So now $C^* = F_0 \supset F_1 \supset \ldots$ with $H_k^* = \operatorname{Im} H^*(F_k) \to H^*(C^*)$ this defines a decreasing filtration of $H^*(C)$...

Theorem 9.2. *Let $p : Y \to X$ be a serre fibration with $\pi_1(X) = 0$ and fiber F. Then there exists a first quadrant cohomological spectral sequence where $E_2^{pq} = H^p(X, H^q(F)) \Rightarrow$ (converging to) $H^*(Y)$.*

We have a map $d_r : E_r^{pq} \to E_r^{p+1, q-r+1}$ with $d_r^2 = 0$, and first quadrant meants p or $q < 0$ implies $E_2^{pq} = 0$.

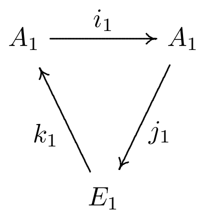
Definition 9.3 (Vanishes for Trivial Reasons)**.** *A homomorphism $f : A \to B$ vanishes for trivial reasons if either $A = 0$ or $B = 0$.*

For an example, lets take $S^1 \to S^{2n+1} \to \mathbb{CP}^n$, the Hopf Fibration. This theorem says that $E_2^{pq} = H^p(\mathbb{CP}^n, H^q(S^1))$ which gives us a pair of horizontal rows which are the cohomology of $\mathbb{CP}^n$ and all others zero. After the first page, the only nonzero entries are the second row, and they are of the form $\ker d_2$, and all maps vanish for trivial reasons, so the spectral sequence stabilizes, that is, $E_3 = E_4 = \dots$. We call the stabilization E_∞.

If the world were just, $\oplus_{p+q=n} E_\infty^{pq} = H^n(Y)$. It's not generally true, but it IS true rationally.

Suppose that A, B are abelian groups with filtrations $A \supset F^*$ and $B \supset G^*$ with $f : A \to B$ compatible with the tibration. Is it true that $grf : grA \to grB$ is an isomorphism implies that $f : A \to B$ is?

10 Lecture 11



Last time, we looked at exact couples like

From this, we got a derived exact couple, and could continue.

Let $A = F_0 A \supset F_1 A \supset \dots$ a filtration, and look at $\mathrm{Gr}_F A = \oplus F_p A / F_{p+1} A$.

Often, we will be sloppy and replaces $F_k A$ by F_k even if it causes confusion.

A map of filtered objects is $f : A \to B$ such that $f(F_k A) \subset F_k B$.

When f is a filtered morphism, it induces a morphism $\mathrm{Gr}(f) : \mathrm{Gr}_F A \to \mathrm{Gr}_F B$.

Lemma 10.1. *If A, B are filtered abelian groups and $f : A \to B$ is a filtered map and the filtrations are bounded (only finitely many terms) then if* $\mathrm{Gr}(f)$ *is an isomorphism, so is f.*

Proof. Duppose that $F_k A = 0$ for all $k \geq N + 1$. Then we know that $F_N A$ is the last nonzero term, and we have isomorphisms for each term of $\mathrm{Gr}_F A$. Then as $F_{n+1} A = 0$, we have $f : F_N A \to F_N A$ is an isomorphism.

Then we have short exact sequences

$$0 \to F_N A \to F_{N-1} A \to F_{N-1} A / F_N A \to 0$$

and are given a morphism of them. We know it is an isomorphism in the right by hypothesis, and have proved that it is in the center, and so by the short five lemma, it is on the left, and so, by induction, we have isomorphisms of all the graded parts. $\square$

Examples of filtered complexes:

If X is a spce, then $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \dots \subset X_n$ (when a CW-complex), we define in teh singular cochain complex a filtration $F_k C^*(X) = \ker\{C^*(X) \to C^*(X_{k+1})\}$.

If $C = (C^{**}, d, \delta)$ is a doubel complex, then $\mathrm{Tot}\, C$ has a filtration given by $F_k \mathrm{Tot}^n C = \oplus_{p+q=n, q \geq k} C^{pq}$.

In both cases, we get an exact couple and a spectral sequence. Recall that first $E_1 = H^*(\mathrm{Gr}_F(C^*))$.

For a double complex, we get $E_2^{pq} = H^q(H^*(C^{p*}, d), \delta)$.

Theorem 10.2 (Serre)**.** *Given a Serre fibration $F \to Y \xrightarrow{p} X$ with $\pi_1(X) = 0$, there exists a spectral sequence with $E_2^{pq} = H^p(X, H^q(F))$ which abuts to $H^{p+q}(Y)$.*

This is in fact a multiplicative spectral sequence, that is, $H^*(Y)$ is an algebra, $H^*(X; H^*(F))$ is an algebra, and the d_r are graded derivation.

So now we continue our study of the cohomology of $U(n)$.

$H^*(U(1), \mathbb{Z}) \simeq \Lambda^*_{\mathbb{Z}}(x), |x| = 1$, that is, $\Lambda^*_{\mathbb{Z}}(x_1, \dots, x_n)$ is the free graded commutative algebra with generators $x_1, \dots, x_n$.

For $\Lambda^*_{\mathbb{Z}}[x]$ with $|x| = 2k$, we have $|x^n| = 2nk$ and so we have $\mathbb{Z}[x]$. For $x = 2k + 1$, we get $\mathbb{Z}[x]/(x^2 = 0)$.

For $\Lambda_{\mathbb{Z}}(x_1, x_2, x_3)$ with $|x_i| = i$. We will list monomials by degree. For $\deg = 0$, we have 1, for $\deg = 1$ we have x_1, for $\deg = 2$ we have x_2, for $\deg = 3$ we have x_3, x_1x_2 for $\deg = 4$, x_2^2, x_1x_3 and in $\deg = 5$, have $x_1x_2^2, x_2x_3$.

We have $U(1) \to U(2) \to U(2)/U(1) = S^3$, and $H^*(S^1 = U(1)) = \Lambda(x_1)$ with $|x_1| = 1$ and $H^*(S^3) = \Lambda(x_3) = \Lambda(x_3)$ with $|x_3| = 3$.

The spectral sequence degenerates and tells us that $H^*(U(2)) = \Lambda_{\mathbb{Z}}(x_1, x_3)$.

Now we look at $U(2) \to U(3) \to S^5$, and we could compute this. However, rather than do it directly, we will use the fact that this is a multiplicative spectral sequence.

We redraw the spectral sequence with generators. $d_5(x_1) = d_5(x_3) = 0$ for trivial reasons, and so $d_5(x_1x_3) = dx_1 \cdot x_3 - x_1 dx_3 = 0$, and so $H^*(U(3)) = \Lambda(x_1, x_3)$. In general, $H^*(U(n)) = \Lambda_{\mathbb{Z}}(x_1, x_2, \dots, x_{2n+1})$.

11 Lecture 12

Some examples of calculations.

ΩS^3. We have a fibration $\Omega S^3 \to PS^3 \to S^3$. PS^3 is contractible. This gives us $E_2^{pq} = H^p(S^3, H^q(\Omega S^3))$. We know that E_∞^{pq} is zero except if $p = q = 1$ when it is $\mathbb{Z}$. We also have $E_2 = E_3$ and $E_4 = E_\infty$. So we must have $H^0(S^3, H^2(\Omega S^3)) \simeq H^3(S^3)$ because the differential is injective (because the term doesn't survive) and surjective. So the cohomology of ΩS^3 is $\mathbb{Z}$ in even degree and 0 in odd degree. These are the same cohomology groups as $\mathbb{CP}^\infty$. Are they homotopic?

And now we want to calculate the cohomology ring. We know $H^*(S^3) = \Lambda_{\mathbb{Z}}(x)$ with $|x| = 3$.

So now replacing groups by their generators, we have the first column as $1, 0, y_2, 0, y_4, \dots$ and the fourth as $x, 0, xy_2, \dots$. So the question is, what is the relationship between y_2^2 and y_4? We know that $d_3 y_4 = xy_2$ (we choose generators such that this is true), and $d_2(y_2^2) = (d_3 y_2) y_2 + y_2 (d_3 y_2) = xy_2 + y_2 x = 2xy_2$, and so $y_2^2 = 2y_4$.

Similarly, there is a $y_6 \in H^6(\Omega S^3)$, and $dy_6 = xy_4$. And $d(y_2^3) = 3y_2^2 dy_2 = 3y_2^2 x = 6y_4$. So $y_2^3 = 6y_6$. So we have the ring $\mathbb{Z}[y, y^2/2, \ldots, y^n/n!, \ldots]$ with $|y| = 2$. This is called the divided power ring.

Now we try with ΩS^2. Then we have $E_2^{pq} = H^p(S^2, H^q(\Omega S^2))$. The first column is $H^i(\Omega S^2)$ and the second column is zero, with the third being $H^2(S^2)$.

Note that after E_2 differentials, all diffs vanish for trivial reasons, and so $E_3 = E_\infty$. This page must have zeroes except in 1,1 and thus we have isomorphisms everywhere else, and so we have $\mathbb{Z}$'s all the way up.

Now we have y_i generates $H^i(\Omega S^2)$, and $dy_1 = x$, $dy_2 = xy_1$ and $y_1^2 = 0$. Note $|x| = 2$. $d(y_1 y_2) = dy_1 y_2 - y_1 dy_2 = xy_2 - y_1 x y_z = xy_2$. So $y_1 y_2 = y_3$. Also, $y_2^2 = 2y_4$, and so $H^*(\Omega S^2) = \Lambda_{\mathbb{Z}}(y_1, y_2, y_1^2/2!, y_2^3/6!, \ldots)$.

There are in fact Homology Spectral Sequences. There is one by Serre:

Let $F \to Y \to X$ be a fibration. Then $E_{pq}^2 = H_p(X, H_q(F)) \Rightarrow H_*(Y)$ with differentials $d_r : E_{pq}^r \to E_{p-q,q+r-1}^r$.

We knwo that $\pi_1(X)/[-,-] = H_1(X)$. for all X.

Step 1: Prove that if $\pi_0(X) = \pi_1(X) = 0$ then $\pi_2(X) = H_2(X)$. Look at $\Omega X \to PX \to X$. By the Serre Spectral Sequence, whatever happens after the first step lasts forever, and so $f : H_2(X) \to H_0(X, H_1(\Omega X)) = H_1(\Omega X) = \pi_1(\Omega X)/[-,-] = \pi_2(X)$. is an isomorphism.

So now we assume that we know $\pi_k(X) = 0$ for $k < n$. Then $\pi_k(X) \simeq H_n(X)$ for all X. Then let Y be a space such that $\pi_k(Y) = 0$ for all $k < n+1$. Apply the Serre sequence to $\Omega Y \to PY \to Y$. Then there is a map $H_{n+1}(Y) \to H_n(\Omega Y)$ which must be an isomorphism and is $\pi_n(\Omega Y) = \pi_{n+1}(Y)$.

Back to the theory:

Let (C, d) be a complex, and $C = F^0 C \supset \ldots$ a filtration of complexes. The spectral seqeunce was constructed from the exact couple $A_1 = \oplus_{p\in\mathbb{Z}} H^*(F^p(C))$ by extending $F^p C = C$ for $p < 0$ with $i : A_1 \to A_1$ the shift induced by $H^*(F^p C) \to H^*(F^{p-1} C)$ and $E_1 = H^*(\mathrm{Gr}_F\, C)$. Suppose the filtration is short. That is, $F^3 C = F^4 C = \ldots = 0$.

Let's start deriving the exact couple.

Then we get $0 \to H^*(F^2 C) \to H^*(F^1 C) \to H^*(F^0 C) \to \ldots$ with all the maps after F^0 isomorphisms. Then A_1 is the direct sum of these.

A_2 is the direct sum of $0 \to iH^*(F^2 C) \to iH^*(F^1 C) \to H^*(F^{-1} C) \to$ and isomorphisms.

Then A_3 is the direct sum of $0 \to iH^*(F^1 C) \to H^*(F^{-1} C) \to$ isos. i is an injection in each of them.

So at A_3 all the maps are inclusions.

Lemma 11.1. *If $A \xrightarrow{i} A \xrightarrow{j} E \xrightarrow{k} A$ is an exact couple with i an inclusion, then the derived couple is the same thing and $E = A/iA$.*

So we have $0 \to A \xrightarrow{i} A \to E \to 0$ a short exact sequence.

So $A_3 = A_4 = \ldots$. So $E_3 = E_\infty = \mathrm{Gr}_F\, H^*(C)$ where F is the filtration on $H^*(C)$ given by $F^p(H^*(C)) = \mathrm{Im}\, H^*(F^P(C)) \to H^*(C)$.

Theorem 11.2. *Given a filtration on* (C, d) *there exists a spectral sequence* $E_1^{pq} = H^{p+q}(F^pC/F^{p+1}C)$ *and if the filtration restricted to each* C^k *is bounded, then this Spectral Sequence* $\Rightarrow H^*(C)$*. That is,* $E_\infty^{pq} = \mathrm{Gr}^p H^{p+q}(C)$*.*

Consider $\mathbb{R}$. Filter $\mathbb{R}$ by $X_n = (-\infty, n)$. Let $C^*\mathbb{R}$ be the singular cochain complex, then $F^pC^* = \ker(C^*(\mathbb{R}) \to C^*(X_n))$. So $E_1^{pq} = H^*(X_n, X_{n-1}) = 0$.

But $H^*(\mathbb{R}) = \mathbb{Z}$ if $* = 0$ and 0 else. So convergence of a spectral sequence isn't literal.

Corollary 11.3. *Suppose* C^*, D^* *are filtered complexes and* $f : C \to D$ *a filtered map. Then if* $f_r : E_r(C) \to E_r(D)$ *is an isomorphism, then* f_s *is an isomorphism for* $s > r$ *and* $f_\infty : E_\infty(C) \simeq E_\infty(D)$ *and* $f : H^*(C) \to H^*(D)$ *is an isomorphism.*

The Serre Spectral Sequence is functorial.

If $F \to Y \to X$ is a fibration, and $f : Z \to X$ a map, then $f^*Y = Z \times_X Y \to Z$ is a fibration with fiber F and f induces a map of Serre sequences. So $f_r : E_r^{pq}(Y \to X) \to E_r^{pq}(f^*Y \to Z)$.

Now we look at $Map(S^1, X)$. The cohomology of this space is useful for constructing closed geodesics on Riemannian manifolds.

12 Lecture 13

So we were looking at $\Omega S^3 \to Map(S^1, S^3) \to S^3$, and using the Serre spectral sequence. For E_2^{pq}, we have alternating $\mathbb{Z}$'s and 0's in the first and fourth columns, and zeros everywhere else, and so $E_2 = E_3$ for trivial reasons. We must compute d_3.

We note that this gives a long exact sequence on homotopy gorups, so we have $\pi_k(\Omega S^3) \to \pi_k(Map(S^1, S^3)) \to \pi_k(S^3) \to \pi_{k-1}(\Omega S^3) \to \ldots$, and also note that this fibration has a section $s(x) = c_x$, the constant map $S^1 \to S^3$, and so this long exact sequence splits.

So we have $0 \to \pi_{k+1}(S^3) \to \pi_k(Map(S^1, S^3)) \to \pi_k(S^3) \to 0$, and so $\pi_k(Map(S^1, S^3)) = \pi_k(S^3) \oplus \pi_{k+1}(S^3)$. In particular, $\pi_2(Map(S^1, S^3)) = \pi_2(S^3) \oplus \pi_3(S^3) = \mathbb{Z}$. And so $H_2(Map(S^1, S^3)) = \mathbb{Z}$, which implies that $H^2(Map(S^1, S^2)) = \mathbb{Z}$. This says that, as $H^2(Map(S^1, S^3)) = \mathbb{Z}$, we have to have a $\mathbb{Z}$ persisting, and so $d_3 = 0$ from $E_3^{2,0} \to E_3^{3,0}$, and so looking at generators, $d_3(x) = 0$, and so $d_3 = 0$ for all parts.

Thus, $E_\infty \simeq \Lambda(x, x^2/2, x^3/3!, \ldots, y)$ where $|x| = 2, |y| = 3$.

For the S^2 case, we have $\mathbb{Z}$ in each dimension and a $\mathbb{Z}$'s and zeros in the third column, but otherwise the same, with generators, $1, x, z, xz, z^2/2!, xz^2/2!, \ldots$ generate the first column, and $y, xy, zy, \ldots$ in the third. So $d_2(x) = 0$, because $H_1 = \mathbb{Z}$, and so it must persist. $d_2(z)$, however, can't be found this way.

We will use the functoriality of the spectral sequence to get this. We have a natural map $\Delta : S^2 \to S^2 \times S^2$. From this and Kunneth Formula, we get a spectral sequence with the same first column, third colum generated by $y \otimes 1, 1 \otimes y$ and products with x, and fourth column similarly with $y \otimes y$. So $d_2(x) = y \otimes 1 - 1 \otimes y$, which goes to zero when we take it back to our original sequence.

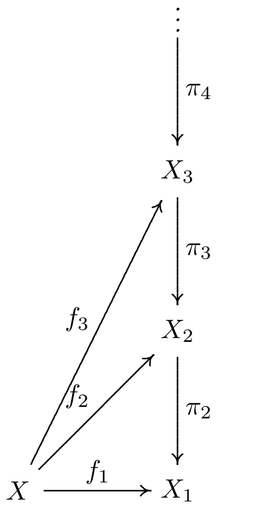
So now we must look at $d_2(x(y \otimes 1)) - y \otimes y$ and $d_2((y \otimes 1)X) = y \otimes y$, and so $x(y \otimes 1) + (1 \otimes y)x$ is in the kernel.

Now $d_2(z) = xy \otimes 1 + x1 \otimes y$. And so, mapping back to the original spectral sequence, we have $z = \Delta^* z$, and so $d_2(z) = d_2(\Delta^* z) = \Delta^* d_2(z) = \Delta^*(xy \otimes 1 + x1 \otimes y) = 2xy$

12.1 Postnikov Tower

This is a "dual decomposition" to a cell decomposition (not exactly, though, but similar role with fibrations that CW complexes do with cofibrations)

Proposition 12.1. *Given a space X, there exists*



such that the π_k are fibrations with fibers $K(\pi_k(X), k)$ with $f_k : X \to X_k$ is $(k-1)$-connected (that is, it is an isomorphism of homotopy groups of dimension up to k)

Proof. To build this, fix n. Then take $X \overset{f_{(n)}}{\to} X_{(n)}$ by killing the homotopy in degree $> n$ by adding cells of degree $> n+1$.

Construct $X_{(n-1)}$ by adding cells of degree n to $X_{(n)}$ to kill homotopy, we have inclusion $X_{(n)} \to X_{(n-1)}$ by the standard construction, we can make this a fibration.

Now we add cells to $X_{(n-1)}$ to kill π_{n-1}, and get $X_{(n-2)}$, etc. □

Proposition 12.2. $H^*(K(\mathbb{Q}, n), \mathbb{Q}) \simeq H^*(K(\mathbb{Z}, n), \mathbb{Q}) \simeq \Lambda_{\mathbb{Q}}(x), |x| = n.$

Proof. $K(\mathbb{Z}, 1) = S^1$, and so $H^*(K(\mathbb{Z}, 1), QQ) = H^*(S^1, \mathbb{Q}) = \Lambda x, |x| = 1$.

Assume that $H^*(K(\mathbb{Z}, n), \mathbb{Q}) = \Lambda_{\mathbb{Q}} x, |x| = n$. Then we have a fibration $\Omega K(\mathbb{Z}, n+1) \simeq K(\mathbb{Z}, n) \to PK(\mathbb{Z}, n) \to K(\mathbb{Z}, n+1)$.

If n is even, then we get a spectral sequence which on the first nontrivial page takes x, the generator of $E^{0,2}$ to y the generator of $H^{n+1}(K(\mathbb{Z}, n), \mathbb{Q})$. So

$dx^k = kx^{k-1}y$, and so we have isomorphisms and get Λy. A similar argument works for n odd. □

Recall the following:

Theorem 12.3. *Given a filtered complex (C, d) with $C = F^0C \supset F^1C \supset \dots$ there exists a spectral sequence with $E_1^{pq} = H^{p+q}(\mathrm{Gr}_F^p(C))$. If the filtration is locally bounded, then it converges to $H^*(C)$.*

Set $A_1^{pq} = H^{p+q}(F^p(C))$ and $E_1^{pq} = H^{p+q}(\mathrm{Gr}_F^p(C))$ with $i : A_1^{pq} \to A_1^{p-1,q-1}$, $j : A_1^{pq} \to E_1^{pq}$ adn $k : E_1^{pq} \to A_1^{p+1,q}$. Then we have $d_r : E_r^{pq} \to E_r^{p+r,q-r+1}$.

Define the filtration by $F^{p+1} = C(X, X^p)$, so then $E_1^{pq} = H^{p+q}(X_p, X_{p-1}) \simeq \oplus\mathbb{Z}$ over e^λ the p-cells for $p + q = p$ and 0 else.

Now let $p : Y \to X$ be a fibration, X path connected. Then filter Y by $Y^k = p^{-1}(X^k)$. The serre spectral sequence is the spectral sequence of the filtration $F^kC^*(Y) = \ker(C^*(Y) \to C^*(Y^k))$, and so $E_1^{pq} = H^{p+q}(Y^p, Y^{p-1})$. Next time, we will computer E_2.

13 Lecture 14

We are going to compute $\pi_4(S^3)$.

Working out the Postnikov tower for S^3, $X_1 = X_2 = 0$, $X_3 = K(\mathbb{Z}, 3)$ and $X_4 = K(G, 4)$ for some $G = \pi_4(S^3)$. This is what we want to calculate.

We take the spectral sequence on homology with

$$E^2_{pq} = H_p(K(\mathbb{Z}, 3), H_q(K(\pi_4(S^3)), 4)))$$

The H_2 page has the form

$$\begin{array}{cccccc}
\pi_4(S^3) & 0 & 0 & \pi_4(S^3) & & \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} & H_4(K(\mathbb{Z},3),\mathbb{Z}) & H_5(K(\mathbb{Z},3),\mathbb{Z})
\end{array}$$

So then $H_k(X_4) \simeq H_k(S^3) = 0$ for $k = 4, 5$, as $X_4 = S^3 \cup_{p \geq 6} e^p_\lambda$.

The only nontrivial differentials are d_4, d_5. Now d_5 must be an isomorphism, so $\pi_4(S^3) = H_5(K(\mathbb{Z}, 3), \mathbb{Z})$, so we must merely compute the homology of $K(\mathbb{Z}, 3)$.

We have $K(\mathbb{Z}, 2) \to PK(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3)$ a fibration. Using the spectral sequence on cohomology given by Serre, we have E_2^{pq}

$$\begin{array}{ccccccc}
\mathbb{Z}x^3 & & & & & & \\
0 & & & & & & \\
\mathbb{Z}x^2 & & & & & & \\
0 & & & & & & \\
\mathbb{Z}x & & & \mathbb{Z}xy & & & \\
0 & & & & & & \\
\mathbb{Z} & 0 & 0 & \mathbb{Z}y & H^4 & H^5 & H^6
\end{array}$$

So $d_3(x) = y$ and so $d_3(x^2) = 2xdx = 2xy$, so $H^4 = 0$.

Now we look at the E_4 level, and to determine it, we analyze $d_3 : \mathbb{Z}xy \to H^6$.

$$\begin{array}{ccccccc} 0 & & & & & & \\ 0 & & & & & & \\ 0 & 0 & 0 & G & & & \\ 0 & 0 & 0 & & & & \\ \mathbb{Z} & 0 & 0 & 0 & 0 & H^5 & H^6 \end{array}$$

If d_3 is the zero homomorphism, then $G = \mathbb{Z}/2$, but this would live to E_∞, and so d_3 is surjective, and so $H^6 = \mathbb{Z}/2\mathbb{Z}$, and so $H_5(K(\mathbb{Z}, 3), \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} = \pi_4(S^3)$.

13.1 Whitehead Towers

Proposition 13.1. *Given a connected CW-complex there exists a diagram with f_i fibrations with a tower ${}_iX \leftarrow K(\pi_i(X), i-1)$ and ${}_iX \to {}_{i-1}X$ with ${}_0X = X$ such that ${}_kX \to X$ induces an ismorphism on π_j for $j > k$ and $\pi_j({}_jX) = 0$ for $j \leq k$.*

Theorem 13.2 (Serre)**.** *$\pi_j(S^n)$ is torsion iff n even, $j \neq 2, 2n-1$ or n odd and $j \neq n$.*

Look at the first step of the Whitehead tower $K(\pi_n(S^n), n-1) \to {}_nX \to S^n$. Then we have $\pi_j({}_nX) \to \pi_j(S^n)$ is an isomorphism for $j > n$.

Last time we calculated $H^*(K(\mathbb{Z}, n-1)) \simeq \Lambda_{\mathbb{Q}}(x)$ with $|x| = n-1$.

So we have

$$\begin{array}{ccccc} \mathbb{Q} & 0 & \dots & 0 & \mathbb{Q} \\ \vdots & & & & \vdots \\ \mathbb{Q} & 0 & \dots & 0 & \mathbb{Q} \end{array}$$

So the only nontrivial differential is $\mathbb{Q}x \to \mathbb{Q}y$, and $dx = y$. If $n-1$ is odd, then $H^{2n-1}({}_nX, \mathbb{Q}) = \mathbb{Q}$ and is zero otherwise.

And so $H_k({}_nX, \mathbb{Q}) = 0$ for all k if n is odd...?

Theorem 13.3. *If $\pi_1 = 0$ and $\pi_k(X) \otimes \mathbb{Q}$ is 0 for $k \leq n$ then $H_k(X, \mathbb{Q}) = 0$ for $k < n$ and $\pi_n(X) \otimes \mathbb{Q} \simeq H_n(X, \mathbb{Q})$*

How to get the Serre Spectral Sequence:

Let $p : Y \to X$ be a Serre fibration with fiber F. Then $p^{-1}(x) \simeq p^{-1}(y)$ for all x, y and assume that $\pi_0(X) = *$. Then we get a map $\pi_1(X, x) \to$ homotopy equivalences of $p^{-1}(x_0)$.

Assume that $\pi_1(X, x_0)$ acts trivially on $H^*(F)$, take X and filter it by its skeleton, and set $Y_k = p^{-1}(X_k)$.

We get a filtration on $C^*(X)$, and so the spectral sequence has $E_1^{pq} = H^{p+q}(F^p/F^{p+1}) = H^{p+q}(Y_p, Y_{p+1})$.

So now let's compute. There exists a commutative diagram

$$\begin{array}{ccc}
H^{p+q}(Y_p, Y_{p-1}) & \xrightarrow{d_1} & H^{p+q+1}(Y_{p+1}, Y_p) \\
\downarrow \wr & & \downarrow \wr \\
H^p(X_p, X_{p-1}) \otimes H^q(F) & \xrightarrow{d_1 \otimes 1} & H^{p+1}(X_{p+1}, X_p) \otimes H^q(F)
\end{array}$$

Where d_1 is the differential in the spectral sequence for filtration of X by skeletons. Look at a p-cell $e^p_\lambda \subset X_p$ and it is attached by a map ψ_λ.

14 Lecture 15

$E_1^{pq} = H^{p+q}(Y_p, Y_{p+1})$, and $d : E^{pq} \to E^{p+q+1}$. Let $F \to Y \to X$ be a fibration. Let $F \to \tilde{D}^n \to D^n$ be a fibration.

Then we want $H^k(\tilde{D}^n, \partial\tilde{D}^n)$. As $S^{n-1} = \partial D^n = D^{n-1}_+ \cup_{S^{n-2}} D^{n-1}_0$, and this is still true if we take $\tilde{D}^n$.

Claim: $H^k(D^j_+, S^{j-1}) = H^{k-1}(D^{j-1}_+, S^{j-2})$.

Look at the triple $(D^j_+, S^{j-1}, D^{j-1}_-)$.

The long exact sequence we get is then $H^k(\tilde{D}^j_+, \tilde{S}^{j-1}) \to H^k(\tilde{D}^j_+, \tilde{D}^{j-1}_-) \to H^k(\tilde{S}^{j-1}, \tilde{D}^{j-1}) \to \ldots$.

So then $H^k(S^{j-1}, D^{j-1}_-) \simeq H^k(D^{j-1}_+, S^{j-2})$ by excising $(D^{j-1}_- \setminus S^{j-2})$.

And so we have $H^k(\tilde{D}^n, \tilde{S}^{n-1}) \simeq \ldots \simeq H^{k-n}(F_{D^0_+})$

So by excision, we get

$$H^{p+q}(Y_p, Y_{p-1}) \simeq \prod_{p-cells} H^{p+q}(\tilde{e}^p_\lambda, \tilde{\partial e}^p_\lambda) \simeq \prod_{p-cells} H^q(F)$$

So the bottom row computes the CW-cohomology of X with coefficients in $H^q(F)$. The face that we don't have to worry about the isomorphisms is because the fundamental group acts trivially.

Thus used the hypothesis that $\pi_1(X)$ acted trivially on $H^*(F)$.

Define a presheaf by taking $U \mapsto H^q(\pi^{-1}(U))$ and we can sheafify to get a sheaf, call it $\underline{H}^q$, locally constant.

There is a more general statement: $E_2^{pq} = H^p(X, \underline{H}^q) \Rightarrow H^{p+q}(Y)$.

A locally constant sheaf is also called a local system.

Theorem 14.1. *The following are equivalent notions:*

1. *A homomorphism $p : \pi_1(X) \to GL(V)$ where V is a k-vector space*
2. *Local systems with stalk V*
3. *Vector bundles whose fiber is V together with a flat connection.*

This is called the Riemann-Hilbert Correspondence.

Now recall the Whitehead Tower.

Theorem 14.2 (Serre's Theorem)**.** *For n odd, $\pi_k(S^n)$ is torsion iff $k \neq n$ and for n even, $\pi_k(S^n)$ is torsion iff $k \neq n, 2n+1$.*

For n odd, we have $S^n \leftarrow X_n \leftarrow K(\pi_n(X), n-1) = K(\mathbb{Z}, n-1)$.Taking the rational cohomology in the spectral sequence, we see that $H^*(X_n, \mathbb{Q}) = 0$.

As $\pi_{n+1}(X_n) \simeq H_{n+1}(X)$, we have that $\pi_{n+1}(X_n) \otimes \mathbb{Q} \simeq \tilde{H}_{n+1}(X_n, \mathbb{Q}) = 0$, and so $\pi_{n+1}(X_n)$ is torsion, and equal to $\pi_{n+1}(S^n)$.

So now $H^*(X_{n+1}, \mathbb{Q}) = 0$, so $\pi_{n+2}(X_{n+1}) = H_{n+2}(X_{n+1})$, which is torsion. Etcetera.

Now, we move on.

Look at the Postnikov tower for S^3. Spectral Sequences give you $H_5(X_4) = \mathbb{Z}/2$, and $H_5(X_4) = \pi_5(X_4) = \pi_5(S^3)$.

We see that in general $\mathbb{Z}/p$ contributes a p-torsion in $\pi_{2p}(S^3)$.

Now we will start with Bundle Theory.

Definition 14.1 (Fiber Bundle)**.** *$\pi : Y \to X$ is a fiber bundle with fiber F if for all $x \in X$, there exists $U \ni x$ and a homeomorphism $\pi^{-1}(U) \to U \times F$ with this map commuting with the projections to X.*

Let (U_α, ϕ_α) be a covering of X such that $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is homeo. We call this a trivializing cover.

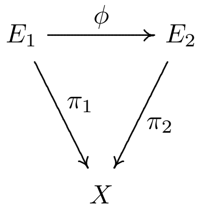
For α, β with $U_\alpha \cap U_\beta \neq \emptyset$, look at $U_\alpha \times F \overset{\phi_\alpha^{-1}}{\to} \pi^{-1}(U_\alpha)$ and restrict, we get $(U_\alpha \cap U_\beta) \times F \overset{\phi_\alpha^{-1}}{\to} \pi^{-1}(U_\alpha \cap U_\beta) \overset{\phi_\beta}{\to} (U_\alpha \cap U_\beta) \times F$. Then we have $\phi_\beta \circ \phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$. This takes $(x, f) \to (x, g_{\beta\alpha}(x) f)$, and so we get a map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to Homeo(F)$, which is a topological group.

15 Lecture 16

Definition 15.1 (Fiber Bundle)**.** *A fiber bundle (E, X, π) with $\pi : E \to X$ with fiber F given $x \in X$ there exists $U \ni x$ and $\phi : \pi^{-1}(U) \to U \times F$ is a homeomorphism over U.*

Definition 15.2 (Section)**.** *A section of a fiber bundle is a continuous map $s : X \to E$ such that $\pi s = 1_X$. Let $\Gamma(X; E)$ be the space of sections.*

Definition 15.3 (Morphism)**.** *A morphism is a commutative diagram*



An isomorphism of bundles is a bundle morphism that is a homeomorphism.

Given a trivializing cover of $\pi : E \to X$, U_α a cover of X and $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ we can form $(U_\alpha \cap U_\beta) \times F \overset{\psi_\beta^{-1}}{\to} \pi_\alpha^{-1}(U_\alpha \cap U_\beta) \overset{\psi_\alpha}{\to} (U_\alpha \cap U_\beta) \times F$, then

$\psi_\alpha\psi_\beta^{-1}(u,f) = (u, g_{\alpha\beta}(u)f)$, and $g_{\alpha\beta}$ satisfy a cocycle relation. $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$, where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to Homeo(F)$. Also require $g_{\alpha\alpha} = 1_F$.

(this is $Map(X, Map(Y,X)) \simeq Map(X \times Y, Z)$)

Conversely, given a cover U_α and a map $g_{\alpha\beta} : U_{\alpha\beta} \to Homeo(F)$ such that $g_{\alpha\alpha} = 1$ and $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$, we can construct a fiber bundle.

Let $E = \coprod_\alpha U_\alpha \times F/\sim$ where $(x,f) \in U_\alpha \times F$ and $x \in U_{\alpha\beta}$ we have $(x,f) \sim (x, g_{\alpha\beta}(x)f)$.

Fact: $\pi : E \to X$ by $\pi(x,f) = x$.

This quotient is a well defined fiber bundle due to the cocycle condition.

Fiber bundles with structure group:

Let G be a topological group. Assume fixed, an action of G on a space F: $F \times G \to F$ such that $(fg_1)g_2 = f(g_1g_2)$.

Definition 15.4 (Fiber Bundle with Structure Group). *A fiber bundle with structure (G,F) is $\pi : E \to X$ such that there exists a cover U_α with $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ such that $g_{\alpha\beta} = \psi_\alpha\psi_\beta^{-1} : U_{\alpha\beta} \to G \subset Homeo(F)$.*

Example 15.1. *Let F be a set and* $\mathrm{Aut}(F)$ *=permutations of F. Then a fiber bundle with structure* $(\mathrm{Aut}\,F, F)$ *is a covering space with fiber F.*

Example 15.2. *If $G = GL_n(\mathbb{R})$ and $F = \mathbb{R}^n$, you get real vector bundles of rank n.*

Example 15.3. *Let $G = Aff(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL_n(\mathbb{R})$ and $F = \mathbb{R}^n$, we get an affine bundle.*

Example 15.4. *Ig $G = GL_n(\mathbb{C})$ and $F = \mathbb{C}^n$ have complex vector bundle.*

Example 15.5. *Let $G = O(n)$ and $F = \mathbb{R}^n$, then we have a vector bundle with inner product.*

If $E \to X$ is an $\mathbb{R}$ vector bundle, an inner product is a map $E \times_X E \to \mathbb{R}$ such that $E_x \times E_x \to \mathbb{R}$ is a positive definite inner product.

Definition 15.5 (Principal Bundle). *Let G be a group, and $F = G$ with action as right multiplication. Bundles with this structure are called principal bundles $\pi : P \to X$.*

All the fibers are G, and so we get an action of G on G freely and properly. These actions fit together and give a free, proper action of G on P, and in fact $P/G \simeq X$.

This is all souces of principal bundles: G acting property and freely on P and $P \xrightarrow{\pi} P/G = X$.

(acting properly means that the map $G \times X \to X \times X$ by $(g,x) \mapsto (gx, x)$ is proper)

Definition 15.6 (Associated Bundle). *Given a principal G bundle $\pi : P \to X$ and α be an action of G on F. Then the associated bundle $P \times_G F \to X$. This is a fiber budnle with structure (G,F).*

Lemma 15.1. *1. If $\pi : P \to X$ a principal bundle and π has a section, then $P = X \times G$.*

2. If $\phi : P_1 \to P_2$ is a morphism of principal bundles, then it is an isomorphism.

Proof. 1. Let s be a section. Define a map $P \to X \times G$ by $p \mapsto (\pi(p), p^{-1}(s(\pi(p))))$. Then there exists a unique $g \in G$ such that $pg = s(\pi(p))$. Define another map $X \times G \to P$ by $(x, g) \mapsto (s(x)g)$. These are inverses.

2. Exercise.

□

This says that the category of principal bundles is a groupoid.

Given a manifold, M, there is a tangent bundle TM. We can define $F(M)_m$, the frame bundle, to be the set of bases for TX_m. This is a principal $GL_n(\mathbb{R})$ bundle.

In fact, $TM = F(M) \times_{GL_n(\mathbb{R})} \mathbb{R}^n$, so it is an associated bundle.

Given $\mathbb{R}^n$, we can form $(\mathbb{R}^n)^*$ and $GL_n(\mathbb{R})$ acts on $(\mathbb{R}^n)^*$ by $\phi \in (\mathbb{R}^n)^*$, $g\phi(v) = \phi(g^{-1}v)$, then $F(M) \times_{GL_n(\mathbb{R})} (\mathbb{R}^n)^* = T^*M$.

$F(M) \times_{GL_n(\mathbb{R})} \bigwedge^k \mathbb{R} = \bigwedge^k TM$.

If we take $F(M) \times_{GL_n(\mathbb{R})} \text{End}(\mathbb{R}^n) = \text{End}(TM)$.

Theorem 15.2. *Given a Lie Group G, there exists a universal principal bundle $EG \xrightarrow{\pi} BG$ such that there exists a one to one correspondence between isomorphism classes of principal bundles over X, $Bun_G X$ and $[X, BG]$ given by $f : X \to BG$ to $f^*EG \to X$.*

16 Lecture 17

Chiral Yoga: If G acts on X on the left, then G acts linearly on $\mathscr{F}(X) = \{f : X \to \mathbb{C}\}$ on the right.

To turn a left action on X into a right action, take $x \cdot g = g^{-1}x$. Then, we have a right action on X and so a left action on $\mathscr{F}(X)$.

A principal bundle gives rise to transition functions on a trivializing cover U_α, $\phi_\alpha : P|_{U_\alpha} \to U_\alpha \times G$ and $g_{\alpha\beta} = \phi_\alpha \phi_\beta^{-1}$.

Now suppose that $\psi : P_1 \to P_2$ is an isomorphism of principal bundles.

Let U_α be a trivializing cover for both. Then $\phi_\alpha^1 : P_1|_{U_\alpha} \to U_\alpha \times G$ and $\phi_\alpha^2 : P_2|_{U_\alpha} \to U_\alpha \times G$.

We now have two sets of transition functions $g_{\alpha\beta}^i = \phi_\alpha^i \phi_\beta^{i,-1}$.

Define $\psi_\alpha : U_\alpha \times G \xrightarrow{\phi_\alpha^{1,-1}} P_1|_{U_\alpha} \xrightarrow{\psi} P_1|_{U_\alpha} \xrightarrow{\phi_\alpha^2} U_\alpha \times G$.

Claim: $\psi_\alpha g_{\alpha\beta}^1 = g_{\alpha\beta}^2 \psi_\beta$. This follows by direct computation.

So now give an open cover U_α of X. Let $Z(U_\alpha; G) = \{g_{\alpha\beta} : U_{\alpha\beta} \to G | g_{\alpha\alpha} = 1, g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}\}$. Define an equivalence relation $g_{\alpha\beta}^1 = g_{\alpha\beta}^2$ if there exist $\psi_\alpha : U_\alpha \to G$ such that $\psi_\alpha g_{\alpha\beta}^1 = g_{\alpha\beta}^2 \psi_\beta$.

Set $H^1(\mathcal{U};G) = Z^1(\mathcal{U};G)/\sim$. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then we have a map $H^1(\mathcal{V},G) \to H^1(\mathcal{U},G)$. Then we define $H^1(X;G) = \varprojlim H^1(\mathcal{U},G)$.

BEWARE! $H^1(X;G)$ is not necessarily a group.

Let $H \subset G$ and P a principal H bundle.

Definition 16.1. *We define* $\operatorname{Ind}_H^G P = P \times_H G$*.*

It turns out that $\operatorname{Ind}_H^G P$ is a principal G-bundle.

Definition 16.2 (Reduction)**.** *If $Q \to X$ is a principal G-bundle, then a reduction of the structure group from G to H is a principel H-bundle such that* $\operatorname{Ind}_H^G P = Q$*.*

Given $H \to G$ we get a map $H^1(X;H) \to H^1(X;G)$.

By definition, TX (a $(GL_n\mathbb{R}, \mathbb{R}^n)$ bundle) has reductions:

1. $GL_n^+(\mathbb{R})$ an orientation
2. $O_n(\mathbb{R})$ a metric
3. $SO_n(\mathbb{R})$ an orientation and a metric
4. $Sp_n(\mathbb{R})$ an almost symplectic structure
5. $GL_n(\mathbb{C})$ an almost complex structure
6. $SL_n(\mathbb{R})$ a volume form
7. $U(n)$ a Hermitian structure

There is a lie group $Spin(n)$ which is the universal cover of $SO(n)$, that is, the nontrivial twofold cover, for $n \geq 3$.

If P is an $SO(n)$ bundle and Q is a $Spin(n)$ bundle such that $Q \times_{Spin(n)} SO(n) = P$ then Q is called a spin structure on P.

(We call the twofold cover of $O(n)$ is called $Pin(n)$.)

Definition 16.3 (Fair cover)**.** *Let $\mathcal{U}$ be a cover of X. We say that $\mathcal{U}$ is fair if there exists a locally finite partition of unity, λ_α.*

A paracompact space always has a fair cover.

We all a prinicple bundle fair if there exists a trivializing fair cover of X.

Every principal bundle on a paracompact space is fair.

Lemma 16.1. *Pullback of a fair principal bundle is fair.*

Lemma 16.2. *If $P \to X \times I$ is a principal bundle and $P|_{X\times[0,1/2]}$ is trivial and $P|_{X\times[1/2,1]}$ is trivial, then P is trivial.*

Proof. We take ϕ, ψ to be trivializations. We must only worry about $X \times \{1/2\} \times G \to P|_{X\times\{1/2\}} \to X \times \{1/2\} \times G$ given by $\phi\psi^{-1} : X \times G \to X \times G$.

This is an automorphism, and so $(x, g) \mapsto (x, \alpha(g))$ where $\alpha : G \to G$ is a continuous map.

Set $\tilde{\psi} : P|_{X\times[1/2,1]} \to X \times [1/2, 1] \times G$ by $\tilde{\psi}(\psi^{-1}(x, t, g)) = (x, t, \alpha(g))$. So then $\tilde{\psi}$ is a trivialization which agrees with φ at $X \times \{1/2\}$, and so we can glue. □

Lemma 16.3. *Let $\pi : P \to X \times I$ be a fair principal bundle. Then there exists a cover U_α of X such that $P|_{U_\alpha\times I}$ is trivial.*

Proof. Let λ_α be a partition of unity on $X \times I$ such that $P|_{\mathrm{supp}^\circ\, \lambda_\alpha}$ is trivial. For any $N = 1, 2, \ldots$ and any multi-index $\vec{\alpha} = (\alpha(1), \ldots, \alpha(N))$ we define $\lambda_{\vec{\alpha}}$ on X by $\lambda_{\vec{\alpha}} = \prod_{1\le i\le N} \min\{\lambda_{\alpha(i)}(x, t)\}$ for $t \in [\frac{i-1}{N}, \frac{i}{N}]$.

$\lambda_{\vec{\alpha}}(x) > 0$ iff $x \times [\frac{i-1}{N}, \frac{i}{N}] \subset \mathrm{supp}^\circ\, \lambda_{\alpha(i)}$ for all $i = 1, \ldots, N$.

We claim that $P|_{\mathrm{supp}^\circ\, \lambda_{\vec{\alpha}}\times I}$ is trivial.

This follows from the previous lemmas.

Thus, P trivializes on $\mathrm{supp}^\circ\, \lambda_{\vec{\alpha}} \times I$ as $\vec{\alpha}$ runs over all N and multi-indices.

To finish, we must ... □

17 Lecture 18

Theorem 17.1. *Let $P \to X\times I$ be a fair principal bundle with $r : X\times I \to X\times I$ by $r(x, t) = (x, 1)$, then $P \simeq r^*P$.*

Remark 17.1. *IF $f : Y \to X$ and $P \to X$ a principal bundle, $Q \to Y$ a principal bundle $f : Y \to X$ and $\tilde{f} : Q \to P$ making it all commute, then $\tilde{f}$ is a G map. We say that $\tilde{f}$ covers f. This is equivalent to having a morphism $Q \to f^*P$.*

Proof. We construct a G-map $\phi : P \to P$ which covers r. Therefore there exists a morphism $P \to r^*P$, and so is an isomorphism.

Let U_α be a cover of X with partitions of unity λ_α such that $P|_{U_\alpha\times I}$ is trivial. $\varphi_\alpha : P|_{U_\alpha\times I} \to U_\alpha \times I \times G$ is a local trivialization.

Alter λ_α to get $\tilde{\lambda}_\alpha$ such that $\max_\alpha \tilde{\lambda}_\alpha(x) = 1$ for all x.

For each α, we define $r_\alpha : X\times I \to X\times I$ by $r_\alpha|_{X\times I\setminus U_\alpha\times I} = 1$ and $r_\alpha(x, t) = (x, \max\{\tilde{\lambda}_\alpha(t), t\})$.

Construct ψ_α covering r_α by $\psi_\alpha(\phi_\alpha^{-1}(x, t, g)) = \phi_\alpha^{-1}(x, r_\alpha g)$.

Well order the index set A for α, and we claim that r is teh compositive of the r_αs over A.

Then $\tilde{\lambda}_\alpha$ is locally finite, and so for any x, there exists $N \ni x$ such that $N \cap U_\alpha \neq 0$ for only finitely many αs.

Then ψ covers r, so we are done. □

Corollary 17.2. *Give a fair $P \to X \times I$ and $i_0, i_1 : X \to X \times I$ the inclusions at 0 and 1, $p : X \times I \to X$ the proejction. THen $P \simeq p^*i_0^*P \simeq p^*i_1^*P$ and $i_0^*P \simeq i_1^*P$.*

Proof. We have $P \simeq r^*P$ from the last theorem.

$i_1 p = r$ and so $P \simeq r^*P \simeq p^*i_1^*P$, and similar for i_1, and we note that $ri_0 = i_1$, and so $i_1^*P \simeq^* r^*P \simeq i_0^*P$. □

Theorem 17.3. *If $P \to X$ is a fair principal bundle and $f_0 \simeq f_1 : Y \to X$, then $f_0^*P \cong f_1^*P$.*

Given this result, we can define a cofunctor $Bun_G : Top \to Sets$ which takes X to isomorphism classes of principal bundles.

1. Bun_G is a homotopy functor.
2. $Bun_G(\bigvee X_\alpha) = \bigvee_\alpha Bun_G(X_\alpha)$.
3. If U, V are two open sets, then $Bun_G(U \cup V) \rightrightarrows Bun_G(U), Bun_G(V) \to Bun_G(U \cap V)$.

These are the hypotheses of the Brown representability theorem:

Theorem 17.4 (Brown Representability)**.** *A homotopy functor $h : Top \to Sets$, which satisfies the three axioms above, then there exists a CW-complex Bh such that $h(X) \simeq [X, Bh]$.*

Thus, there is a representing space for Bun_G.

This approach is insufficient though. We want to understand the classifying space for Bun_G.

Definition 17.1. *The space that represents Bun_G is called BG. This is only determined up to homotopy.*

BG has a universal bundle on it, given by $1 \in [BG, BG]$. This element corresponds to a bundle $EG \to BG$.

Theorem 17.5. *There exists a universal bundle $EG \to BG$ for principal G-bundles. That is, given $P \to X$ a principal G-bundle, there exists a unique homotopy class of maps $f : X \to BG$ such that $f^*EG \cong P$.*

Next time, we will look at the recognition principle and then perform a general construction of $EG \to BG$

Theorem 17.6 (Recognition Principle)**.** *If you find a principal G-bundle $Q \to B$ where Q is contractible as a space, then $B = BG$ and $Q = EG$.*

Everything needs to be "fair"

Example 17.1. *For $G = \Gamma$ a discrete group, then $\widetilde{K(\Gamma, 1)}$ is the universal bundle over $K(\Gamma, 1)$.*

Example 17.2. *If $G = S^1 = U(1)$, then we get $BU(1) = \mathbb{CP}^\infty$ and $EU(1) = S^\infty$.*

Example 17.3. *For $U(n)$, we get that $BU(n)$ is the grassmanian of n-planes in $\mathbb{C}^\infty$.*

Definition 17.2 (Characteristic Class)**.** *A characteristic class is a natural transformation $c : Bun_G \to H^*$.*

So now, given $EG \to BG$, we know what all the characteristic classes are: given c a characteristic class, apply it to EG, then $c(EG) \in H^*(BG)$ and given a class $\gamma \in H^*(BG)$, define a characterisc class by $c_X(P) = f^*\gamma$ where $P \cong f^*EG$.

Note that $H^*(\mathbb{CP}) \simeq \mathbb{Z}[c]$ with $|c| = 2$.

As $\mathbb{CP} = BU(1)$, this c is called the first Chern class for line bundles.

This isn't all there is to know, however. We need to know how to calculate them and, in general, there will be more than one, and so we need to know the relations between them.

18 Lecture 19

Remark 18.1. *$\pi : P \to X$ a principal G-bundle. The $\pi^*P \to P$ is a principal bundle.*

$\pi^*P = \{(p_1, p_2)|\pi(p_1) = \pi(p_2)\}$.

There is an obvious section $s(p) = (p, p)$, and so $\pi^*P \cong P \times G$ by $(p_1, p_2) \mapsto (p_1, p_1^{-1}p_2)$

Definition 18.1 (n-universal)**.** *A principal G-bundle $\pi : P \to X$ is called n-universal if for any CW complex A of dimension $\leq n$, with principal G-bundle $Q \to A$. Then if $B \subset A$ is a subcomplex and $\phi : Q|_B \to P$ then ϕ extends to $\tilde{\phi} : Q \to P$.*

Theorem 18.1. *If $P \to X$ is n-universal, then for any CW-complex A of dimension $< n$, the map $[A, X] \to \mathrm{Bun}_G(A)$ by $\{f : A \to X\} \mapsto f^*P$ is an isomorphism of sets.*

Proof. If $f_0 \simeq f_1$, then $f_0^*P \cong f_1^*P$ from last time, and so the map is well defined.

Also note that n-universality implies that given any such A and a principal bundle $Q \to A$, there exists a map $\tilde{\phi} : Q \to P$ (by taking $B = \emptyset$).

Thus, the induced map $\phi : A \to X$ taking $Q \cong \phi^*P$ exists, and so we have surjectivity.

Injectivity: Suppose that $\phi_0, \phi_1 : A \to X$ such that $\phi_0^*P \cong \phi_1^*P$. Then we have $\psi : \phi_0^*P \cong \phi_1^*P$ and $\tilde{\phi}_0 : \phi_0^*P \to P$ and $\tilde{\phi}_1 : \phi_1^*P \to P$.

On $A \times I$ consider $\phi_0^*P \times I \to A \times I$. Define $\tilde{\tilde{\phi}} = \begin{cases} \tilde{\phi}_0 & t = 0 \\ \tilde{\phi}_1^*\psi & t = 1 \end{cases}$. And so $\tilde{\tilde{\phi}}$ extends to a map on all of $\phi_0^*P \times I$ and its induced map on $A \times I$ is a homotopy between ϕ_0 and ϕ_1. □

Theorem 18.2. *$\pi : P \to X$ is n-universal if and only if $\pi_k(P) = 0$ for $k = 0, \dots, n-1$.*

Proof. ⇒: Let $\phi : S^{k-1} \to P$. Then $S^{k-1} \times G \to P \times G \simeq \pi^* P \to P$ is a map of principal bundles. By n-universality, this extends to $\tilde{\phi} : B^k \times G \to P \times G$, but $H : B^k \to O$ by $(b, g) \mapsto \tilde{\phi}(b, e)$ an $H|_{S^{k-1}} = \phi_1 \Rightarrow [\phi] = 0$ in $\pi_k(P)$.

⇐: Assume $\pi_k(P) = 0$ for $k = 0, \ldots, n-1$.

Take $B^k \times G$ and a map $\phi : S^{k-1} \times G \to P$ by $f : S^{k-1} \to P$ via $f(s) = \phi(s, e)$. Since $\pi_k(P) = 0$, this extends to a map $\tilde{f} : B^k \to P$. Then $\tilde{\phi} : B^k \times G \to P$ is defined by the composition $B^k \times G \stackrel{\tilde{f} \times 1}{\to} P \times G \cong \pi^* P \to P$ is an extension.

Finally, extension to cell complexes is done cell by cell using what we proved. □

Theorem 18.3. *$\pi : P \to X$ is universal if $\pi_k(P) = 0$ for all k. $[A, X] \cong \mathrm{Bun}_G A$ for all CW A.*

General Construction of $EG \to BG$.

Introduction to Simplicial Techniques:

Define a category Δ with object set $\{[0], [1], [2], \ldots\}$ and $\Delta([m], [n]) = \{$non decreasing maps from $\{0, \ldots, m\} \to \{0, \ldots, n\}$ are the morphisms. So $\Delta = Ord$ is the Simplicial Category.

This category is coming from the geometry of simplices. There exist special morphisms $\partial_i : [n] \to [n+1]$ the injective maps, $i = 0, \ldots, n+1$, and $s_j : [n] \to [n-1]$ the surjective maps, of which there are n.

So geometrically, there are two maps of the zero simplex into the one simplex, and only one backwards, etc.

We call that ∂_i the face maps and the s_j the degeneracies. Note that ∂_i, s_j generate Δ.

Let $\mathcal{C}$ be a category. A simplicial $\mathcal{C}$ object is a contravariant functor from Δ to $\mathcal{C}$, $X : \Delta \to \mathcal{C}$. Define $X_n = X([n])$.

Example 18.1.
1. *Simplicial Sets*
2. *Simplicial Abelian Groups*
3. *Simplicial Groups*
4. *Simplicial Spaces*
5. *Simplicial Simplicial Sets*
6. *Simplicial Schemes*

Example 18.2.
1. *A simplicial complex gives a simplicial set. Let K be a simplicial complex. Order the vertices arbitrarily. Define $X_n = \{(v_0, \ldots, v_n)$ the vertices in K with $v_0 \leq v_1 \leq \ldots \leq v_n | \{v_0, \ldots, v_n\}$ is a simplex in $K\}$. Then $\partial_i(v_0, \ldots, v_n) = (v_0, \ldots, \hat{v}_i, \ldots, v_n)$ and $s_j(v_0, \ldots, v_n) = (v_0, \ldots, v_j, v_j, \ldots, v_n)$.*
2. *LetX be a topological space. Define $Sing_n X = \{f : \Delta_n, X | f$ continuous$\}$, then $\partial_i = \partial_i^*$ and $s_j = s_j^*$ simplicial sets.*

Let A_n be a simplicial abelian group. $\partial_i : A_n \to A_{n-1}$. Define $\partial : A_n \to A_{n-1}$ by $\sum_{i=0}^{n}(-1)^i\partial_i$ a homomorphism. Simplicial identities then imply that $\partial^2 = 0$.

This gives a functor $\natural : sAb \to$ Complexes of Abelian groups. And so we can take the homology of simplicial sets.

Theorem 18.4 (Dold-Kan)**.** *$\natural$ is an equivalence of homotopy categories.*

Let A_n be a simplicial abelian group, and forget the group structure. So you get a simplicial set. To a simplicial set X_n associate $|X|$ the geometric realization, then $\pi_k(|A|) = H_k(A^\natural)$.

19 Lecture 20

$E_2^{pq} \Rightarrow H^*$ converging means: there exists a filtration on H^* such that E_∞ is isomorphic to $\operatorname{gr} H^*$.

So we know that $\mathrm{Bun}_G(X) \simeq [X, BG]$.

Proposition 19.1. *Suppose $X = \sum Y$. Then* $\mathrm{Bun}_G(X) \cong [Y, G]$.

Proof. $\sum Y = C^+Y \cup_Y C^-Y$, and $C^\pm Y = [0,1] \times Y/(0,y) \sim *$. And so $C^\pm Y \simeq *$, and $C^+Y \cap C^-Y = Y$.

Take a bundle P on X. Restrict to C^+Y and C^-Y where it becomes trivial. And so the transition functions are $g : C^+Y \cap C^-Y \to G$. Let $g^1, g^2 : Y \to G$ be two such transition functions. If the associated principal bundles P_1 and P_2 are isomorphic, then there exist $h^\pm : C^\pm Y \to G$ such that $g^2h^+ = h^-g^1$, that is, $g^2 = h^-g^1h^+$.

Define $H : Y \times I \to G$ by $H(y,t) = h^-(y,t)g^1(y)h^+(y,t)$ with $H(y,0) = g_1$ and $H(y,1) = g^2$.

And so $[Y, G] \cong [\sum Y, BG] \cong [Y, \Omega BG]$. □

First let $[G, G] \cong [\sum G, BG] \cong [G, \Omega BG]$ by $1 \mapsto i : G \to \Omega BG$.

Proposition 19.2. *$i : G \to \Omega BG$ is a homotopy equivalent. We say that BG is a delooping of G.*

Definition 19.1 (∞-loop Space)**.** *An ∞-loop space is a sequence E_n of spaces together with homotopy equivelence (or homeomorphism) $E_n \to \Omega E_{n+1}$ to each n.*

Take an infinity loop space. Then $E^n(X) = [X, E_n] \cong [X, \Omega E^{n+1}] \cong [X, \Omega^2 E^{n+2}]$. Then the E^n satisfy the axioms of a homology theory, except the dimension axiom.

Example 19.1. *If $E^n = K(\mathbb{Z}, n)$, then this gives representable cohomology. (Good Cohomology for Crappy Spaces)*

Theorem 19.3. *Let $U = U(\infty) = \lim_{n\to\infty} U(n)$. Then $\Omega U \cong \mathbb{Z} \times BU$. Also $\Omega(\mathbb{Z} \times BU) = \Omega BU \cong U$.*

Corollary 19.4. *$\mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U, \dots$ is an ∞-loop space. The Bott Spectrum gives K-theory.*

With O instead of U, we get $\Omega^8(\mathbb{Z} \times BO) = \mathbb{Z} \times BO$.
So now what is $\mathrm{Bun}_G(S^n)$? It is $\mathrm{Bun}_G(\sum S^{n-1}) = [S^{n-1}, G] = \pi_{n-1}(G)$.
Back to Simplicial stuff.

Definition 19.2 (Simplicial Object)**.** *A simplicial $\mathcal{C}$-object is a collection of objects in $\mathcal{C}$, X_n, indexed by the non-negative integers together with maps $\partial_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ for $0 \le i \le n$ satisfying $\partial_i \partial_j = \partial_{j-1}\partial_i$ for $i < j$, $s_i s_j = s_{j+1} s_i$ for $i \le j$, $\partial_i s_j = s_{j-1}\partial_i$ if $i < j$, $\partial_j s_j = \mathrm{id} = \partial_{j+1} s_j$, and $\partial_i s_j = s_j \partial_{i-1}$ for $i > j+1$.*

So last time we defined a functor $Sing : Top \to sSets$, and now we define $|-| : sSets \to Top$ by taking X_* a simplicial set. Then look at $\coprod_n (X_n \times \Delta^n)/\sim$ via $(x, \partial_i t) \sim (\partial_i x, t)$ for $x \in X_n, t \in \Delta_{n-1}$, and the same thing with the degeneracies.

Remark 19.1. *Geometric Realization and Sing are adjoint functors.*

And so $sSets(X_*, Sing_* Y) = Top(|X_*|, Y)$. And so there is a way to set of homotopy theory of sSets such that the homotopy there and on Top are equivalent.
Start with $Y \mapsto Sing_* Y \mapsto |Sing Y|$ is a weak homotopy equivalence.
This is a completely functorial way to go from a space ot a CW-complex weakly equivalent to it.

19.1 Simplicial Spaces

Geometric realization: Let X_* be a simplicial space, then $|X_*| = \coprod X_n \times \Delta_n/\sim$ (use its topology)
Let Y be a topological space and $\mathcal{U}$ a cover.

Definition 19.3. *$\check{C}(\mathcal{U}) = \coprod_\alpha U_{\alpha_0} \leftleftarrows \coprod U_{\alpha_0 \alpha_1} \dots$. with the maps being ∂_i*

Fact: $|\check{C}(\mathcal{U})_*|$ is homotopy equivalent to Y.

20 Lecture 21

Let X, Y be simplicial sets, then $(X \times Y)_n = X_n \times Y_n$ where $\partial_i(x, y) = (\partial_i x, \partial_i y)$ adn similarly for s_j. Then fact: $|X \times Y| \cong |X| \times |Y|$.

Example 20.1. *Let $X = Y$ be two 0-simplices joined by a 1-simplex. Then work out $X \times Y$.*

Given X, Y sSets construct a bisimplicial set $(\Delta^0 \times \Delta^0) \to Sets$ then $(X \times Y)_{n,m} = X_n \times Y_m \to X_n \times Y_{m-1}$ by $1 \times \partial_j$, and similarly.
There are four ways to get a space from a bisimplicial set.

1. $\coprod Z_{n,m} \times \Delta^n \times \Delta^m / \sim$

2. Z_n is simplicial set for each n, $n \mapsto |Z_{n,0}|$ simplicial space, etc, then geometrically realize.

3. $|m \mapsto |Z_{0,m}||$

4. From $Z_{n,m}$ form $(\Delta Z)_n = Z_{n,n}$, and look at $|\Delta Z|$.

Theorem 20.1. *All of these are homeomorphic.*

Let $\mathcal{C}$ be a small category. From $\mathcal{C}$ we define a simplicial set where $B_0\mathcal{C}$ are the object, $B_1\mathcal{C}$ are morphisms in $\mathcal{C}$, $B_2\mathcal{C}$ are sequences $x_0 \stackrel{\phi_0}{\to} x_1 \stackrel{\phi_1}{\to} x_2$, and similarly for $B_k\mathcal{C}$.

We have ∂_i given by forgetting an object and composing morphisms. The s_j are given by replacing x_j with $x_j \stackrel{\mathrm{id}}{\to} x_j$.

This is a simplicial set.

Define $B\mathcal{C} = |B_*\mathcal{C}|$. This is the classifying space of the category.

Embellishment: If $\mathcal{C}$ is a topological category, that is, $ob\mathcal{C}$ and $\hom\mathcal{C}$ are topological and composition is continuous, then $B_*\mathcal{C}$ is a simplicial space.

Example 20.2. *1. If Γ is a group, then Γ is a category with 1 object and whose automorphism group is Γ. Then $B\Gamma$ is $K(\Gamma, 1)$.*

2. If G is a Lie group, then it is a topological category and BG is really BG.

3. Let X be a simplicial complex, and define $\mathcal{C}$ whose objects are the simplices of X and $\sigma \to \tau$ when σ is a face of τ. Then $|B\mathcal{C}| \cong X$ with the simplicial structure of the Barycentric subdivision

If $\mathcal{C}$ and $\mathcal{D}$ are two small categories and $F, G : \mathcal{C} \to \mathcal{D}$ are functors, we get simplicial maps $BF, BG : B\mathcal{C} \to B\mathcal{D}$.

Proposition 20.2. *If N is a natural transformation from F to G, then $BF \simeq BG$.*

Proof. For all $x \in \mathcal{C}$, there exists a morphism in $\mathcal{D}$, $N_x : F(x) \to G(x)$, such that if $f : x \to y$ in $\mathcal{C}$, then the appropriate diagram commutes. □

Define $\mathcal{I}$ to be the category with two objects, $\{0\}, \{1\}$ and one nonidentity morphism $\{0\} \to \{1\}$, then $B\mathcal{I} \cong I = [0, 1]$.

Now, note that a natural transformation $N : F \Rightarrow G$ is the same as a functor $H : \mathcal{C} \times \mathcal{I} \to \mathcal{D}$ which satisfies the usual homotopy conditions.

Let X be a topological space, define $\mathcal{E}X$ to be teh topological category with object the points of X and morphisms in $X \times X$.

Proposition 20.3. *$B\mathcal{E}X$ is contractible.*

Proof. Pick a point $x_0 \in X$. Define $F : \mathcal{E}X \to Cat\{x_0\}$ and G the embedding in the opposite direction.

Then FG is the identity functor and GF takes everything to x_0 and 1_{x_0}.

Claim: there exists N from GF to $1_{\mathcal{E}X}$ by $N_x : GF(x) \to x$, then N_x is the unique morphism from $x_0 \to x$. And so, $\mathcal{E}X$ is contractible. □

Take G a topological group, and let BG be the classifying space from above.

Theorem 20.4. *BG is BG, the classifying space for principal G-bundles.*

Proof. for $\mathcal{E}G$. Then look at $B\mathcal{E}G$. Now $B\mathcal{E}G$ is contractible. G acts on the simpicial space $\mathcal{E}G$ freely and properly. Thus $B\mathcal{E}G/G$ wil be an avatar of BG. □

Corollary 20.5. *If Γ is a group, then $H^*(B\Gamma;\mathbb{Z}) \cong H^*(\Gamma;\mathbb{Z})$, where the left is singular and the right is group cohomology.*

Proof. Recall that $B\Gamma$ is a simplicial set, and $\hom(B\Gamma, \mathbb{Z}) \cong \hom_{\mathbb{Z}}(\mathbb{Z}B\Gamma, \mathbb{Z})$ $(\mathbb{Z}B\Gamma, \sum(-1)^i\partial_i)$ is a chain complex. Computing $H_*(B\Gamma)$ we see th $\hom_{\mathbb{Z}}(\mathbb{Z}B\Gamma, \mathbb{Z})$ computes $H^*(B\gamma, \mathbb{Z})$, but $\hom(B\Gamma, \mathbb{Z})$ is isomorphic to the cochain complex computing the group cohomology. □

If Γ is a group, then $B\Gamma$ is a simplicial set, and all the appropriate ∂_i are group homomorphisms iff Γ is abelian.

Let $\mathbb{A}$ be an abelian group. BA is a simplicial abelian group. Take $B(BA)$, this is a bisimplicial abelian group. $BA = K(A, 1)$, $BBA = K(A, 2)$, etc.

Take ΔBBA, then B^iA is a simplicial abelian group, and we can continue.

21 Lecture 22

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22 Lecture 23

Characteristic Classes

Thom Class and Isomorphism

Theorem 22.1. *For $E^k/\mathbb{R} \to X$ an oriented rank k real vector bundle, then there exists a unique $\Phi \in H^k_{CV}(E, \mathbb{Z})$ such that $\Phi|_{E_x} \in H^k_c(E_x, \mathbb{Z})$ is the canonical class defined by the orientation.*

$H^*_{CV}(E, \mathbb{Z}) = H^*(E, \dot{E}; \mathbb{Z})$ where $\dot{E} = E \setminus z(X)$ the zero section, and is isomorphic to $H^*(B_E, S_E)$, where B_E is the ball bundle and S_E the sphere bundle.

$\Phi : H^i(X) \to H^{i+k}_{CV}(E) \cong H^{i+k}(E, \dot{E}) \cong H^{i+k}(B_E, S_E)$ where $\Phi(x) = \pi^*(x) \cup \Phi_E$ is an isomorphism.

Definition 22.1 (Euler Class)**.** $e(E) = z^*\Phi_E \in H^k(X, \mathbb{Z})$.

Proposition 22.2. *1. $f : Y \to X$, then $f^*e(E) = e(f^*E)$, and $(f^*E)_y = E_{f(y)}$, so $\Phi_{\tilde{f}^*E} = \tilde{f}^*\Phi_E$, where $\tilde{f} : f^*E \to E$.*

2. *Let $-E$ be E with the opposite orientation, then $e(-E) = -e(E)$.*

3. *$c(E \oplus F) = e(E) \cup e(F)$.*

4. *If k is odd, then $2e(E) = 0$, so $e(-E) = -e(E) = e(f^*E) = f^*e(E) = e(E)$ when f is orientation reversing diffeo along fibers.*

5. *If $\pi : E \to X$ has a nowhere zero section, then $e(E) = 0$.*

Theorem 22.3 (Leray-Hirsch)**.** *Let $\pi : Y \to X$ be a fiber bundle and suppose that $\alpha_1, \ldots, \alpha_k \in H^*(Y)$ such that $\alpha_i|_{H^*(\pi^{-1}(x))}$ is a basis of the cohomology. Then $H^*(Y)$ is a free $H^*(X)$-module with basis $\alpha_1, \ldots, \alpha_k$.*

This can be proved easily using spectral sequences, for example.

If V is a complex vector space of dimension n, then $\mathbb{P}(V) \cong \mathbb{CP}^{n-1}$. The Tautological bundle is $\tau = \{(\ell, v) \in \mathbb{P}(V) \times V | v \in \ell\}$. We get an exact sequence $0 \to \tau \to \mathbb{P}(V) \times V \to Q \to 0$.

τ^* is the hyperplane bundle.

$H^*(\mathbb{P}(V)) = \mathbb{Z}[x]/x^n$, setting $x = c_1(\tau^*)$ and $|x| = 2$.

Lemma 22.4. *$c_1(\tau^*)$ is a generator of $H^*(\mathbb{P}(V))$.*

Let $\tau : \mathscr{V}^n \to X$ be a $\mathbb{C}$-vector bundle of rank n. So $g_{\alpha\beta} : U_{\alpha\beta} \to GL_n\mathbb{C}$. We have $GL_n\mathbb{C} \to PGL_n\mathbb{C} = \mathrm{Aut}(\mathbb{CP}^{n-1})$

We can then associate a bundle $\mathbb{P}(\mathscr{V}) \to X$, where $\mathbb{P}(\mathscr{V}) = \mathscr{P} \times_{PGL_n} \mathbb{CP}^{n-1}$, where $\mathscr{P}$ is a principal $PGL_n\mathbb{C}$-bundle.

On $\mathbb{P}(\mathscr{V})$ we have $\tau_{\mathscr{V}}$, a tautological line bundle. $p \in \mathbb{P}(\mathscr{V})$, $\pi(p) \in X$, so p is a line in $\mathscr{V}_{\pi(p)}$. Then $\tau_{\mathscr{V}} = \{(x, \ell, v) \in X \times \mathbb{P}(\mathscr{V}_x) \times V_x | v \in \ell\}$.

So now take $c_1(\tau^*_{\mathscr{V}}) \in H^2(\mathbb{P}(V))$. Then powers of this are global classes with the proeprty that when restricted to a fiber $\mathbb{P}(\mathscr{V}_x)$, $c_1(\tau^*_{\mathscr{V}})|_{\mathbb{P}(\mathscr{V}_*)}$ is isomorphic to $c_1(\tau^*_{\mathscr{V}_x})$ where $\tau_{\mathscr{V}}|_{\mathscr{V}_x} = \tau$ in $\mathbb{P}(\mathscr{V}_x)$.

By Leray-Hirsch, $H^*(\mathbb{P}(V)) \cong H^*(X)[1, c_1(\tau^*_{\mathscr{V}}) = x, x^2, \ldots, x^{n-1}]$ is a basis as a module.

So there exists a relation $x^n + c_1(\mathscr{V})x^{n-1} + c_2(\mathscr{V})x^{n-2} + \ldots + c_n(\mathscr{V}) = 0$. The coefficients $c_i \in H^{2i}(X)$ are called the i^{th} Chern Classes of $\mathscr{V}$. And in fact, $H^*(\mathbb{P}(\mathscr{V})) \cong H^*(X)[x]/(x^n + c_1x^{n-1} + \ldots + c_n)$.

Theorem 22.5. *Let $c(\mathscr{V}) = 1 + c_1(\mathscr{V}) + \ldots + c_n(\mathscr{V}) \in H^{ev}(X)$ be called the total chern class. It has the following properties:*

1. *c, c_k are natural.*

2. *$c_k(\mathscr{V}) = 0$ if $k > \mathrm{rank}\, \mathscr{V}$.*

3. *$c(V \oplus W) = c(V) \cup c(W)$. (This is the Whitney Sum Formula)*

4. *If $\mathscr{V}$ has rank n, then $c_n(\mathscr{V}) = e(\mathscr{V}^{2n}/\mathbb{R})$.*

5. *If L_1, L_2 are line bundles, then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.*

Proof. We prove 5 first. $g^1_{\alpha\beta}, g^2_{\alpha\beta}$ denote the transitions for L_1, L_2 respectively on the same covering. Define $\tilde{g}^i_{\alpha\beta}$ to be the log of $g^i_{\alpha\beta}$. So then $c_1(L_1) = (\delta\tilde{g}^1)_{\alpha\beta\gamma} = \tilde{g}_{\beta\gamma} - \tilde{g}_{\alpha\gamma} + \tilde{g}_{\alpha\beta}$.

The transition functions for $L_1 \otimes L_2$ are the products, and so $\tilde{g}^1 + \tilde{g}^2 = \tilde{g^1 g^2}$, and so we're done.

1 and 2 are simple also. The rest is next time. □

23 Lecture 24

Let $\pi : E^k/\mathbb{R} \to X$ be an oriented vector bundle.

$\Phi : H^i(X) \to H^{i+k}(E, \dot{E})$ by $a \mapsto \pi^* a \cup \Phi_E$. Look at the LES of $\dot{E} \to E$.

$\to H^i(E, \dot{E}) \to H^i(E) \to H^i(\dot{E}) \to H^{i+1}(E, \dot{E}) \to \ldots$ We have maps identifying $H^i(E) \cong H^i(X)$ and $H^i(E, \dot{E}) \cong H^{i-k}(X)$.

Gysin Sequence: $\to H^j(X) \to H^{j+k}(X) \to H^{j+k}(\dot{E}) \to H^{j+1}(X) \to$

Look at the Gysin Sequence for $\tau \to \mathbb{CP}^n$. Then we have $\to H^1(\dot{\tau}) \to H^0(\mathbb{CP}^n) \to H^2(\mathbb{CP}^n) \to H^2(\dot{\tau}) \to H^1(\mathbb{CP}^n) \to \ldots$. $\dot{\tau} = \mathbb{C}^{n+1} \setminus \{0\} \simeq S^{2n+1}$, and so $H^2(\dot{\tau}) = H^1(\dot{\tau}) = 0$ if $n \neq 0$. And so, $\cup e(\tau)$ is an isomorphism on $H^*(\mathbb{CP}^n)$ until $H^{2n-1}(\mathbb{CP}^n) \to H^{2n+1}(\mathbb{CP}^n)$.

$H^*(\mathbb{CP}^n) \cong \mathbb{Z}[e(\tau)]$.

Remark 23.1. *If L is a line bundle, then $L^*_x = \hom_{\mathbb{C}}(L_x, \mathbb{C}) = \bar{L}_x$. So then $(\bar{L})/\mathbb{R} = -(L/\mathbb{R})$. So $c_1(L \otimes L^*) = c_1(\hom(L, L)) = c_1(1) = 0$, and so $c_1(L^*) = -c_1(L)$.*

Theorem 23.1 (Characterization of Chern Classes)**.**

1. *$c(E) \in H^*(X)$*
2. *$c(f^*E) = f^*c(E)$ when $f : Y \to X$ and $\pi : E/\mathbb{C} \to X$.*
3. *Normalization: $c(\tau^*) = 1 + x$ in $\mathbb{CP}^n$.*
4. *Whitney Sum Formula: $c(E \oplus F) = c(E) \cup c(F)$*

Additionally, these properties characterize the chern clases.

More properties

1. $c_j(E^k) = 0$ for $j > k$
2. $c_k(E^k) = e(E/\mathbb{R})$
3. $H^*(BGL_n(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n]$ where $|c_i| = 2i$, where $c_i = c_i(EGL_n(\mathbb{C}) \times_{GL_n\mathbb{C}} \mathbb{C}^n)$.

Everything but the Whitney formula follows easily from the naturality of c_1 and $\mathbb{P}$.

We now prove Whitney.

Let $i : E \to E \oplus F$ take $e_x \mapsto (e_x, 0_x)$. Then $\mathbb{P}$ is a functor (at the level of spaces) and so $i' : \mathbb{P}(E) \to \mathbb{P}(E \oplus F)$. Define $p : E \oplus F \to F$ the projection. Then define $p^- : \mathbb{P}(E\oplus F)\setminus i'(\mathbb{P}(E)) \to \mathbb{P}(F)$. Let $j : \mathbb{P}(E\oplus F)\setminus i'(\mathbb{P}(E)) \to \mathbb{P}(E\oplus F)$ the inclusion.

Facts: $(i')^*\tau_{E\oplus F} = \tau_E$, $j^*(\tau_{E\oplus F}) = (p')^*\tau_F$.

Define $\alpha = \sum_{i=0}^{e=\operatorname{rank} E} \pi^*_{E\oplus F}(c_i(E))x^{e-i}_{E\oplus F} \in H^{2e}(\mathbb{P}(E\oplus F))$.

$\beta = \sum_{\ell=0}^{f} \pi^*_{E\oplus F}(c_\ell(F))x^{f-\ell}_{E\oplus F} \in H^{2f}(\mathbb{P}(E\oplus F))$.

$(i')^*\alpha = \sum_{i=0}^{e}(i')^*\pi^*_{E\oplus F}(c_i(E)) \cup (i')^*x^{e-i}_{E\oplus F} = \sum_{i=0}^{e} \pi^*_E c_i(E) \cup x^{e-i}_E = 0$.

So now $j^*\beta = \sum_{k=0}^{f} j^*\pi^*_{E\oplus F}(c_k(F))j^*x^{f-k}_{E\oplus F} = \sum_{k=0}^{f}(p')^*\pi^*_F(c_k(F))(p')^*x^{f-k}_F = p'\sum_{k=0}^{f} \pi^*_F(c_k(F))x^{f-k}_F = 0$.

By the Long exact sequence from $\mathbb{P}(E\oplus F), \mathbb{P}(E\oplus F)\setminus i'(\mathbb{P}(E))$

$H^*(\mathbb{P}(E\oplus F), \mathbb{P}(E\oplus F)\setminus i'(\mathbb{P}(E))) \xrightarrow{\ell} H^*(\mathbb{P}(E\oplus F)) \xrightarrow{j^*} H^*(\mathbb{P}(E\oplus F)\setminus i'(\mathbb{P}(E)))$.

Then $\alpha\cup\beta = \alpha\cup\ell(\beta') = (i')^*\alpha\cup\beta' = 0$. So $0 = \alpha\cup\beta$. This is the defining relation for $c(E\oplus F)$ (if you work it out carefully)

Now we move on to the splitting principle

If $0 \to E \to F \to L \to 0$ is an exact sequence of vector bundles, then $F \cong E\oplus L$.

This is a consequence of continuous partitions of unity.

Corollary 23.2. *Let X be compact. There is an equivalence of categories between finite dimensional vector bundles on X and finitely generated projective modules over $C(X)$.*

Proposition 23.3. *Given $E^k/\mathbb{C} \to X$ a vector bundle, there exists $F(E) \to X$ a fiber bundle such that $H^*(X) \to H^*(F(E))$ by $\pi^*_{F(E)}$ is injective and $\pi^*_{F(E)}E = L_1 \oplus \ldots \oplus L_k$ where L_i are complex line bundles.*

Corollary 23.4 (Splitting Principle)**.** *In order to check any identity on Chern classes, it suffices to check it on sums of line bundles.*

Check: $c_k(E) = e(E/\mathbb{R})$. Then let $F(E) \to X$ be a splitting space.

Then $\pi^*(c_k(E)) = c_k(\pi^*E) = c_k(L_1\oplus\ldots\oplus L_k) = c_1(L_1)\cup\ldots\cup c_1(L_k) = e(L_1)\cup\ldots\cup e(L_k) = e(L_1\oplus\ldots\oplus L_k) = e(\pi^*E) = \pi^*e(E)$. As π^* is injective, $c_k(E) = e(E)$.

24 Lecture 25

Proposition 24.1. *Given $\pi : E^k/\mathbb{C} \to X$, there exists a fiber bundle $p : F(E) \to X$ such that $p^*E \cong L_1\oplus\ldots\oplus L_k$ and $p^* : H^*(X) \to H^*(F(E))$ is injective.*

Proof. Form $\mathbb{P}(E) \to X$. On $\mathbb{P}(E)$, we have τ_E, which fits into $0 \to \tau_E \to p^*E \to E_1 \to 0$ and $H^*(\mathbb{P}(E)) = H^*(X)[1, x, x^2, \ldots, x^{k-1}]$ where $x = c_1(\tau^*_E)$.

the map $p^* : H^*(X) \to H^*(\mathbb{P}(E))$ is injective, and $p^*E \cong \tau_E\oplus E_1$. Apply the same construction to E_1. Then induction proves the result. ☐

To see that $c(1_n) = 1$, note that $(1_n \to X) = f^*(1_n \to pt)$ which has no cohomology to pull back.

We know that $c(\tau^*) = 1 + x$, with $x \in H^2(\mathbb{CP}^n)$ a generator.

$T\mathbb{CP}^n$: $T_\ell \mathbb{CP}^n = \hom_{\mathbb{C}}(\ell, \tau_\ell^\perp)$ and so $T\mathbb{CP}^n = \hom(\tau, \tau^\perp)$. We have $0 \to \tau \to 1_{n+1} \to \tau^\perp \to 0$. Tensor this with τ^*, and we get $0 \to \tau^* \otimes \tau \to \tau^* \otimes 1_{n+1} \to \tau^* \otimes \tau^\perp \to 0$. This is just $0 \to 1_1 \to \oplus_{i=1}^{n+1} \tau^* \to T\mathbb{CP}^n \to 0$. So noncanonically, $\tau^* \oplus \ldots \oplus \tau^* \cong 1_1 \oplus T\mathbb{CP}^n$. And so $c(\tau^* \oplus \ldots \oplus \tau^*) = c(1_1 \oplus \mathbb{CP}^n)$. This is $c(\tau^*)^{n+1} = c(1_1)c(T\mathbb{CP}^n)$, so $c(\tau^*)^{n+1} = c(T\mathbb{CP}^n)$

And so, $(1+x)^{n+1} = c(T\mathbb{CP}^n)$. And so, $c(T\mathbb{CP}^n) = 1 + (n+1)x + \binom{n+1}{2}x^2 + \ldots + \binom{n+1}{n}x^n$. So then $c_i(T\mathbb{CP}^n) = \binom{n+1}{i}x^i$.

Now note that τ^* is a holomorphic (even algebraic) line bundle. $\Gamma_{hol}(\mathbb{CP}^n, (\tau^*)^{\otimes k})$ is the set of homogeneous polynomial of degree k in $z_0, \ldots, z_n$

If you have a homogeneous polynomial p of degree k, this defines a setion of $(\tau^*)^{\otimes k}$. The zeros of this section as a subset of $\mathbb{CP}^n$ are the zeroes of p in $\mathbb{CP}^n$.

Take a section s and suppose that H_k is the zeroes of s which intersects transversely to the zero section z of $(\tau^*)^{\otimes k}$.

Since s and z are transverse, the normal bundle to H_k in $\mathbb{CP}^n$ is $(\tau^*)^{\otimes k}|_{H_k}$, we will denote this by ν. We have $0 \to TH_k \to T\mathbb{CP}^n|_{H_k} \to \nu \to 0$, and so $c(\mathbb{TCP}^n|_{H_k}) = c(TH_k \oplus \nu) = c(TH_k)c(\nu) = c(TH_k)(1 + kx)$.

And so $c(TH_k) = \frac{(1+x)^{n+1}|_{H_k}}{1+kx|_{H_k}}$.

Curve=Riemann Surface. LEt p be a degree k homogeneous polynomial with $p^{-1}(0)$ smooth.

$c(TH_k)|_{H_k} = \frac{(1+x)^3}{1+kx} = \frac{1+3x}{1+kx}$, and that is $1 + (3-k)x$. We often call the Chern class of the tangent bundle to a manifold the Chern class of the manifold.

Euler characteristic of H_k is determined by g, and is $\chi(H_k) = 2 - 2g$.

On the other hand

$$\begin{aligned}
\chi(H_k) &= \langle e(TH_k), [H_k] \rangle \\
&= \langle c_1(TH_k), [H_l] \rangle \\
&= \langle (3-k)x, [H_k] \rangle \\
&= \langle (3-k)x, [\mathbb{CP}^2] \cap kx \rangle \\
&= \langle (3-k)x \cup kx, [\mathbb{CP}^2] \rangle \\
&= \langle k(3-k)x^2, [\mathbb{CP}^2] \rangle \\
&= k(3-k)
\end{aligned}$$

And so $2 - 2g = k(3-k)$, and so $g = \frac{(k-1)(k-2)}{2}$.

Quartic in $\mathbb{CP}^3$

This defines $K \subset \mathbb{CP}^3$ a Kummer Surface (or K3).

$c(TK) = \frac{(1+x)^4}{1+4x} = 1 + 6x^2$. So $c_1(TK) = 0$.

For real vector bundles, $E^k/\mathbb{R}$, we define the k^{th} Pontrjagin class $p_k(E) = (-1)^k c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C})$.

Example: $p_1(TK) = c_2(TK \otimes_{\mathbb{R}} \mathbb{C}) = c_2(TK \oplus TK) = c_2(TK) + c_2(TK) = 12x^2$.

Let $\pi : E^k/\mathbb{C} \to X$ be a vector bundle. Think of E formally as $L_1 \oplus \ldots \oplus L_k$. So then $c(E_k) = c(L_1 \oplus \ldots \oplus L_k) = c(L_1)\ldots c(L_k) = \prod(1 + c_1(L_i))$, and so $c_k(E)$ is the k^{th} elementary symmetric function in the $c_1(L_i) = x_i$'s.

Thus, if an expression in the x_i's is symmetric, you can express it in terms of Chern classes.

We define the L-polynomial to be $L(E^k/\mathbb{C}) \equiv \prod_{i=1}^k \frac{\sqrt{x_i}}{\tanh\sqrt{x_i}}$, the Todd polynomial $Td(E) = \prod_{i=1}^k \frac{x_i}{1-e^{x_i}}$, and $ch(E) = \sum_{i=1}^k e^{x_i}$, the Chern character.

$ch(E \oplus F) = ch(E) + ch(F)$, $ch(E \otimes F) = ch(E)ch(F)$.

If M is a compact oriented manifold of dimension $4k$, then $sign(M)$is given by taking $H^{2k}(M) \times H^{2k}(M) \to H^{4k}(M) = \mathbb{Z}$ a symmetric nondegerate bilinear pairing. Diagonalize and take the number of + eigenvalues minus the - eigenvalues.

If $M = \partial W$, and W compact, then $sign(M) = 0$. $sign(\mathbb{CP}^n) = 1$.

Theorem 24.2 (Hirzebruch's Signiture Theorem)**.** $sign(M) = \langle L(TM \otimes_{\mathbb{R}} \mathbb{C}), [M]\rangle$

In dimension 4, $L = p_1/3$, where p_1 is the first Pontrjagin class.

$sign(K) = \langle p_1(TK)/3, K\rangle = \langle 4x^2, [K]\rangle = \langle 4x^2, \mathbb{CP}^3 \cap [4x]\rangle = 16$.

Theorem 24.3 (Lichnerowisz)**.** *K has no metric of positive scalar curvature.*