1 Schnell - Deligne’s Theorem on Abelian Varieties, Part I

Theorem 1.1 (Deligne). On an abelian variety, all Hodge classes are absolutely Hodge.

The proof breaks up into two parts:
1. reduce to the case of CM abelian varieties
2. Deal with CM case.

We’ll deal with step 1 today.

Recall, in the case of weight 1:

Definition 1.2 (CM field). A CM field is a number field $E$ of the form $E = F[t]/(t^2 - f)$ where $f \in F$ and $F$ is totally real and under all embeddings $F \subset \mathbb{R}$, $f$ is negative.

Definition 1.3 (CM abelian variety). An abelian variety is CM if there exists a CM field $E \subset \text{End}(A) \otimes \mathbb{Q}$ such that $\dim_E H^1(A, \mathbb{Q}) = 1$

This implies that $[E : \mathbb{Q}] = \dim_C H^1(A, \mathbb{Q}) = 2 \dim A$.

There is a nice criterion $MT(A) = MT(H^1(A, \mathbb{Q}))$ (MT means Mumford-Tate group)

Proposition 1.4. If $A$ is simple, then $A$ is CM if and only if $MT(A)$ is abelian.

Proposition 1.5. Given any abelian variety $A$ and a Hodge class $\alpha$ on $A$, there exists a family $A \rightarrow B$ of abelian varieties with $B$ irreducible and quasi-projective such that there exists $0 \in B$ with $A_0 \cong A$ and the Hodge locus of $\alpha$ is $B$, and there is $t \in B$ where $A_t$ is CM.

Proof. Choose a polarization $Q$ and let $G = \text{Aut}(H^1(A, \mathbb{Q}), Q)$ and $M = MT(A)$ the smallest $\mathbb{Q}$-subgroup whose $\mathbb{R}$-points contain the image of $\phi : \mathbb{S}^1 \rightarrow G(\mathbb{R})$.

Abelian varieties of the same kind, along with a choice of basis for $H^1(A, \mathbb{Z})$, are parameterized by the period domain $D = G(\mathbb{R})/K$. 

Note: points of $D$ are classes of $gH$ in terms of $\phi$. $\phi_{g,H} = g\phi g^{-1}$.

Main idea: family comes from the Mumford-Tate domain: $D_{\phi} = M(\mathbb{R})/M(\mathbb{R}) \cap K \subset D$.

This should have the properties that for all Hodge structures $H' \in D_{\phi}$,

1. $MT(H') \subset M$
2. any Hodge tensor for $A$ is a Hodge tensor for $H'$
3. $\phi_{H'} = g\phi g^{-1}$ for $g \in M(\mathbb{R})$.

Finding CM points corresponds to finding points with abelian MT. $\phi(\mathbb{S}^1) \subset M(\mathbb{R})$ contained in some maximal $\mathbb{R}$-torus $T_0$, and we can show that for $\xi_0 \in m_{\mathbb{R}}$ generic, $T_0$ is the stabilizer of $\xi_0$.

Nearby, there exists $\xi \in m_{\mathbb{R}}$ close to $\xi_0$, then if $T$ is the stabilizer of $\xi$, it is a $\mathbb{Q}$-torus. There exists $g \in M(\mathbb{R})$ such that $\xi = g\xi_0g^{-1}$, and $g\phi g^{-1}$ has image in $T$. Then $MT(H_{g\phi g^{-1}}) \subset T$ is abelian.

Problem: family over quasi-proj base, not $D_{\phi}$. Solution: Fix an $N >> 0$ and use a level $N$ structure.

Define $\mathcal{M}_{g,Q,N}$ to be the moduli space of abelian varieties of dimension $g$ with polarization $Q$ and level $N$ structure (a basis of the $N$-torsion points) and let $A_{g,Q,N} \to \mathcal{M}_{g,Q,N}$ the universal family. OUr replacement for $D_{\phi}$ is to let $B \subset \mathcal{M}_{g,Q,N}$ be the Hodge locus of the Hodge tensors for $H^1(A, \mathbb{C})$ defining $MT(A)$.

$B$ is algebraic by CDK, and finite etale over $\Gamma \backslash D_{\phi}$. In this case, things are ok.

Proof that (for $A$ simple), $MT(A)$ abelian implies $A$ is CM.

We start with the fact that $E = \text{End}(A) \otimes \mathbb{Q}$ is a division algebra, since $A$ is simple. It is also the set of $\mathbb{Q}$-endomorphisms that commute with $MT(A) = M$.

So we know that $M$ is abelian, and thus it acts on $H^1(A, \mathbb{C})$, and we can write $H^1(A, \mathbb{C}) = \oplus_{\chi} H^1(A, \mathbb{C})_{\chi}$ for characters, and thus $E \otimes \mathbb{C} = \oplus_{\chi} \text{End} H^1(A, \mathbb{C})_{\chi}$.

And so $\dim_{\mathbb{Q}} E \geq \dim H^1(A, \mathbb{Q}) = 2\dim A$ is bounded above by $2\dim A$. So $2\dim A = \dim \mathbb{Q}E$ and thus $E$ is a commutative field, so $\dim E H^1(A, \mathbb{Q}) = 1$.

Now, use the Rosati involution $\phi \mapsto \phi^t$ on $E$, and $Q(\phi h_1, h_2) = Q(h_1, \phi^t h_2)$, and $F$ the fixed field. We claim that $[E : F] = 2$ and $F$ is totally real.

We have that $F = \mathbb{Q}(\phi)$, with $\phi = \phi^t$ and take the minimal polynomial. Then $\lambda_j$, the roots, are the eigenvalues of the action of $\phi$ on $H^1(A, \mathbb{Q})$, and if we set $\lambda = \lambda_j$, and $\phi$ acts on $H^1(A, \mathbb{C})$ preserving $H^{1,0} = H^{0,1}$, there exists the roots, are the eigenvalues of the action of $\phi$ on $H^1(A, \mathbb{Q})$, and if we set $\lambda = \lambda_j$, and $\phi$ acts on $H^1(A, \mathbb{C})$ preserving $H^{1,0} = H^{0,1}$, there exists $h \in H^{1,0}$ with $\phi(h) = \lambda h$, $\phi(\bar{h}) = \lambda \bar{h}$. Look at $Q(\phi h, \bar{h}) = Q(h, \phi \bar{h})$, this is $\lambda Q(h, \bar{h}) = \lambda Q(h, \bar{h})$ and so $\lambda = \bar{\lambda}$, so $\lambda \in \mathbb{R}$.

### 2 Kerr - Deligne’s Theorem on Abelian Varieties, Part II

Let $A$ is a CM abelian variety, that is, an abelian variety such that $MT(H^1(A))$ is abelian.
Now, if \( t \in H^{2p}(A^{an}, \mathbb{Q}) \cap F^p H^{2p}_{dH}(A) \) and \( \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}) \) then we want to show that \( t^\sigma \in F^p H^{2p}_{dH}(A^\sigma) \) lies in \( H^{2p}(A^{an,\sigma}, \mathbb{Q}) \).

Let \( E/\mathbb{Q} \) be a CM field of degree \( 2e \) such that \( E \) is totally imaginary and there exists \( p \in \text{Gal}(E/\mathbb{Q}) \) with \( p^2 = \text{id}, \phi \circ p = \phi \) for all \( \phi \in \text{hom}(E, \mathbb{C}) \).

Now, take \( F \) to be the totally real fixed field, and \( \xi \) such that \( E = F(\xi) \), and \( \xi^2 \in F \) and \( \sqrt{-1} \phi_i(\xi) > 0 \) for \( i = 1, \ldots, e \) with \( \phi_i \in \text{hom}(E, \mathbb{C}) \) generated by \( \Phi = \{ \phi_1, \ldots, \phi_e, \bar{\phi}_1, \ldots, \bar{\phi}_e \} \). We call \((E, \Phi)\) the CM type of \( E \).

Now, consider \( A/\mathbb{C} \) an abelian variety with \( E \rightarrow \text{End}(A) \otimes \mathbb{Q} = \mathcal{E} \). Then \( V = H^1(A, \mathbb{Q}) \) is an \( E \)-vector space of even dimension \( d \) and \( \dim A = ed = D \).

Now, \( V \) is self-dual, and so \( E \) acts on \( V^\vee \) and we have a natural quotient map \( \wedge^d V^\vee \rightarrow \wedge^d V^\vee \), and the dual is an inclusion defined over \( E \).

\( E \) is a \( \mathbb{Q} \)-vector space of dimension \( 2e \) and it acts on \( E \otimes \mathbb{Q} \mathbb{C} = \oplus_{\phi \in \text{hom}(E, \mathbb{C})} E^\phi \), adn similarly for \( V \), adn have

\[
\begin{align*}
\wedge^d V & \cong \bigoplus \Lambda^d V_{\phi_i} \cong (\wedge^d V)_C \\
\Lambda^d V & \rightarrow \Lambda^d V
\end{align*}
\]

The HS on \( V \) may be viewed as \( \phi : \mathbb{U} \rightarrow \text{GL}(V) \) taking \( zz \) to the \( \mathbb{C} \)-linear endomorphism of multiplication by \( z^{1-0} \) on \( V^{1,0} \) and \( z^{0-1} \) on \( V^{0,1} \) and this must commute with \( v(E) \).

Therefore, \( V_{\phi_i} = (V_{\phi_i} \cap V^{1,0}) \oplus (V_{\phi_i} \cap V^{0,1}) = V^{1,0}_{\phi_i} \oplus V^{0,1}_{\phi_i} \), and are of dimension \( a_i \) and \( b_i \) with \( a_i + b_i = d_i \).

So the Hodge type of \( \wedge^d V_{\phi_i} \cong \bigwedge_{\mathbb{C}} a_i V^{1,0}_{\phi_i} \oplus \bigwedge_{\mathbb{C}} b_i V^{0,1}_{\phi_i} \) is \((a_i, b_i)\).

Conclusion: If \( \dim(V^{1,0}) = d/2 \) for each \( i = 1, \ldots, 2e \), then \( \bigwedge_{\mathbb{C}}^d V \subset \bigwedge_{\mathbb{Q}}^d V \) consists of Hodge classes (the Weil classes).

If \( A_0 \) is an abelian variety of dimension \( d/2 \) and \( A = A_0 \otimes \mathbb{Q} E = A_0 \times \cdots \times A_0 \) \( 2e \) times, this is then \( \mathbb{C}^{d/2} \otimes \mathbb{C}^{2e} / \Lambda \otimes \mathcal{E}_E \). Let \( V = H^1(A, \mathbb{Q}) \), this is just \( H^1(A_0, \mathbb{Q}) \otimes \mathcal{E} \), and so taking \( E \) to act on the factor of \( E \), we get \( V_{\phi_i} \cong V_{\phi_i} \otimes \mathbb{C}_{\phi_i} \cong V_{i,\mathbb{C}} \).

This gives us that \( \bigwedge_{\mathbb{C}}^d V_{\phi_i} \cong \bigwedge_{\mathbb{C}}^d V_{i,\mathbb{C}} = H^d(A_0, \mathbb{C}) \cong H^{d/2,d/2}(A_0) \).

Moreover, \( \text{Aut}(\mathbb{C}) \) changes neither the product structure on \( A \), the endomorphisms (which are defined by cycles in \( A \times A \)) nor the class of \([p]\) on \( A_0 \). Thus, \( \bigwedge_{\mathbb{C}}^d V \) in this cases consists of absolute Hodge classes.

Now, think of \( V \) as a fixed \( \mathbb{Q} \)-vector space of dimension \( D \) with nondegenerate alternating form \( Q : V \times V \rightarrow \mathbb{Q} \).

Let \( \phi \) be any weight 1 Hodge structure on \( V \) polarized by \( Q \) and \( E \rightarrow \text{End}(V, \phi) \) an isomorphism (in such a way that \( Q \) gives \( V^{1,0,\phi} \) and \( V^{0,1,\phi} \)). We impose the condition that \( \dim V^{1,0,\phi} = d/2 \) for all \( i \).

Then there exists a unique \( E \)-Hermitian form \( \psi : V \times V \rightarrow E \) with \( Q = \text{tr}_{E/\mathbb{Q}}(\mathcal{E} \cdot \psi) \) and \( \phi \) stabilizes \( \psi \) and commutes with \( i(E) \). Hence, \( M_\phi \subset \text{Aut}_E V \cap \text{Sp}(V, \mathbb{Q}) = \text{Res}_{E/\mathbb{Q}} U_E(V, \psi) \) and \( X = M_\phi(\mathbb{R})^+ \subset h^D \) is a MT domain which
precisely classifies the abelian varieties (or HS's) satisfying the above conditions which are precisely that the HS for which $\bigwedge^d V \subset \bigwedge^d V$ consists of Hodge classes.

Now, $\mathcal{A} \rightarrow \Gamma \backslash X$ a torsion free congruence subgroup is by the Baily-Borel theorem a quasi-projective algebraic variety parameterizing such $\mathcal{A}$.

Applying Principle B again leads to

**Theorem 2.1.** *Weil classes on "Veil algebraic varieties" are absolute Hodge*

The rest of Deligne’s proof: Let $\mathcal{M}$ be cut out by $H_{q_A'}$ and $\tilde{\mathcal{M}}$ be cut out by $AH_{q_A'}$ (the Hodge and absolute Hodge tensors) then

**Theorem 2.2** (Principle A). *If a tensor $t \in T^{k,\ell}H^1(A, \mathbb{Q})$ is fixed by $\tilde{\mathcal{M}}$, then it is absolute Hodge.*

For CM abelian varieties, Deligne shows that $\tilde{\mathcal{M}} \supseteq \mathcal{M}$ is an equality by producing enough absolute Hodge classes to push $\tilde{\mathcal{M}}$ inside $\mathcal{M}$. He does this by looking at endomorphisms of the CM field, $A_{\sigma \Phi} \rightarrow A_{\Pi}$ and Weil Hodge classes.

This is dense on $\prod_{\Phi_i} A_{\Phi_i}$. 