Complex manifolds, Kahler metrics, differential and harmonic forms

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1 Lecture 1

Definition 1.1 (Complex Manifold). *A complex manifold is a manifold with coordinates holomorphic on $\mathbb{C}^n$ rather than C^∞ on $\mathbb{R}^n$.*

What is the difference betwene holomorphic and C^∞?

From the PDE point of view, they must satisfy the Cauchy-Riemann equations: $f(z) = u + iv$ is holomorphic if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

The fact that this causes the function f to be analytic(holomorphic) is what gives the theory a very algebraic flavor.

Example 1.2 (Sphere). *We get a map $S^2 \setminus \{N\} \to \mathbb{C}$ and another $S^2 \setminus \{S\} \to \mathbb{C}$ by stereographic projection. The map from $S^2 \setminus \{N, S\}$ to itself that takes the image of the first to the image of the second is $\frac{1}{z}$ on $\mathbb{C}^*$, which is holomorphic. So what are the holomorphic functions? Any holomorphic function gives a bounded entire function by removing a point, and so must be constant! This is in fact true for holomorphic functions on compact connected complex manifolds.*

Theorem 1.3. *The only global holomorphic functions on a compact connected complex manifold are the constants.*

Corollary 1.4. *There are no compact complex submanifolds on $\mathbb{C}^n$ of dimension greater than 0.*

Example 1.5 (Projective Space). *Set $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/\sim$ where $z' \sim z$ iff there exists $\lambda \in \mathbb{C}^*$ such that $z' = \lambda z$. We put charts on it by looking at $U_i = \{[z] \in \mathbb{P}^n | z_i \neq 0\}$.*

Question 1. *When is a compact complex manifold a submanifold of $\mathbb{P}^n$?*

We will also be interested in

Example 1.6 (Grassmannians). *$G(k, n)$ is the space of k-dimensional subspaces of $\mathbb{C}^n$.*

Two manifolds may be C^∞ diffeomorphic but not holomorphic.

Example 1.7 (Torus)**.** *As a C^∞ manifold, a torus is just $S^1 \times S^1$. However, it can have many complex structures. To get one, view it as $\mathbb{C}/\Lambda$ where Λ is a lattice isomorphic to $\mathbb{Z}^2$. There are many different lattices, and so we can get many complex structures this way. It is not hard to see that they are distinct.*

The theory of complex manifolds splits into compact and noncompact, we're only going to look at the compact complex manifolds.

So what do we do? We have no holomorphic functions. A function can be viewed as being a map $M \to M \times \mathbb{C}$ by $x \mapsto (x, f(x))$. To get something like a function, we can change $M \times \mathbb{C}$ to something that is only locally this product, that is, a line bundle.

Definition 1.8 (Line Bundle)**.** *A holomorphic line bundle is $E \xrightarrow{\pi} M$ where M And E are complex manifolds and π is holomorphic such that the fiber over each point is isomorrphic to $\mathbb{C}$ and it is locally biholomorphic to $U_\alpha \times \mathbb{C}$.*

Taking a trivializing open cover U_α, we get maps $U_\alpha \cap U_\beta \to \mathrm{GL}(1, \mathbb{C})$ (or n in the case of more general vector bundles) which we will call $g_{\alpha\beta}$, and they will satisfy $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$, these are called the gluing data.

We can get another bundle using ${}^t g_{\alpha\beta}^{-1}$, and these will give the dual bundle, whose fibers are naturally the dual of the original, and call it E^*. Similarly, we can take E, F and put the transition functions in blocks, and we get $E \oplus F$. For line bundles, the transition functions are numbers, not matrices, so their product is another invertible function, so we get a line bundle $E \otimes F$.

Example 1.9 (Universal Line over $\mathbb{P}^n$)**.** *We set $\mathcal{T} = \{([z], v) | v \in [z]\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1}\}$ and its projection map to $\mathbb{P}^n$ makes it a line bundle with transition maps z_i/z_j.*

So, we replace the notion of a global function with a global section of a line bundle $E \to M$, which is a map $M \to E$ such that $x \in E_x$. This means that on each U_α we have a function f_α and $f_\alpha = g_{\alpha\beta} f_\beta$.

Let $P(x_0, \ldots, z_n)$ be a homogeneous polynomial of degree k. Then on U_i, set $f_i = P/z_i^k$, these form a global section of $(\mathcal{T}^*)^k$. But, if we remove the dual, there are no global sections.

2 Lecture 2

Example 1.16 and A.1

In mathematics, we often try to reduce problems to linear algebra, because linear algebra is something that we understand very well.

If we set $g_{\alpha\beta} = D(\phi_\alpha \circ \phi_\beta^{-1}) \in \mathrm{GL}(n, \mathbb{R})$. In fact, we get $\left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R}) \right\}$, and these are the changes of coordinates preserving the matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

On a complex manifold M, it is possible to define a linear map $J_p : T_p M \to T_p M$ by $J_p(\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$ and $J_p(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$.

Then $J_p^2 = -id$ and so $J^2 + I = 0$.

Now, take $[T_pM]_{\mathbb{C}} = T_pM \otimes_{\mathbb{R}} \mathbb{C} = T_pM \oplus iT_pM$. Call T_p' and T_p'', the parts with eigenvalue i and $-i$.

We can actually, for a complex manifold M, make T_pM into a complex vector space, by defining $(a+ib)*v = av + bJ_pv$. And then, this vector space is isomorphic to $T_p'M$ by taking v to $v - iJv$.

So what is the tangent bundle of $\mathbb{P}^n$? Let $L(t) \in \mathbb{P}^n$ a curve. Fix $v \in L(p) = L$. So the curves don't depend on motion inside the line, thus we end up with $T(\mathbb{P}^n) = \hom_{\mathbb{C}}(\mathcal{T}, \mathbb{C}^{n+1}/\mathcal{T})$, and by changing the lines to subspaces, the same is true of Grassmannians.

Now, we write $[T_p^*M]_{\mathbb{C}} = T_p^{1,0}(M) \oplus T_p^{0,1}(M)$ where the first has basis $dz_i = dx_i + idy_i$ and the second has basis $d\bar{z}_i = dx_i - idy_i$. Then, $df = \sum_j df(\frac{\partial}{\partial z_j})dz_j + \sum_j df(\frac{\partial}{\partial \bar{z}_j} d\bar{z}_j$.

We now set $A^{p,q}(M)$ to be the space of forms whose terms have $dz_I \wedge d\bar{z}_{\bar{J}}$ where $|I| = p$ and $|\bar{J}| = q$. Then the exterior derivative maps $d : A^{p,q}(M) \to A^{p+1,q}(M) \oplus A^{p,q+1}(M)$. So $d = \partial + \bar{\partial}$, both of which square to zero, and also $\partial\bar{\partial} + \bar{\partial}\partial = 0$.

We then have that $A^k(M, \mathbb{C}) = \oplus_{p+q=k} A^{p,q}(M)$. The Dolbeault cohomology, then, is $H_{\bar{\partial}}^{p,q}(M)$ is the cohomology of $\bar{\partial}$.

Inside of $A^{p,0}(M)$ is $\Omega^p(M)$, which is the collection of the forms with holomorphic, not merely C^∞, coefficients.

Now we note that the Dolbealt cohomology is NOT diffeomorphism invariant. It depends on the complex structure of the manifold in question.

3 Lecture 3

Let V be a real vector space along with an operator $J^2 = -I$. This makes it a complex vector space. We can also say $V_{\mathbb{C}} = V \oplus iV = V' \oplus V''$ where V' is the i-eigenspace and V'' the $-i$-eigenspace. We write $V_{\mathbb{C}}^* = V^{1,0} \oplus V^{0,1}$ along with J^*, which are the i and $-i$ eigenspaces. Then $\bigwedge^r V_{\mathbb{C}}^* = \bigoplus \bigwedge^{p,q}$ with $p + q = r$. So $\bigwedge^2 V_{\mathbb{C}}^* = \bigwedge^{2,0} \oplus \bigwedge^{1,1} \oplus \bigwedge^{0,2}$, which are $-1, 1, -1$ eigenspaces respectively.

So now, we have $d : A^{p,q}(M) \to A^{p+1,q} \oplus A^{p,q+1}$ and for each p we get a complex $0 \to \Omega^p(U) \to A^{p,0}(U) \to A^{p,1} \to \ldots$ which is exact for a small enough U (more precisely, exact as a complex of sheaves.

For $p = n = \dim M$, then $\Omega^n(U) = f dz_1 \wedge \ldots \wedge dz_n$ with f holomorphic.

Proposition 3.1. *The Following are equivalent*

1. *A symmetric bilinear form $B : V \times V \to \mathbb{R}$ such that $B(Ju, Jv) = B(u, v)$*
2. *An alternating form $\omega : V \times V \to \mathbb{R}$such that $\omega(Ju, Jv) = \omega(u, v)$*
3. *A hermitian form $H : (V, J) \times (V, J) \to \mathbb{C}$ with $H(Jv, Ju) = H(v, u)$.*

Now, we move to manifolds. Every complex manifold has a positive definite Hermitian structure on the holomorphic tangent bundle, which is equivalent to every complex manifold has a Riemannian metric compatible with J.

By this, we mean that on $(U, z_1, \ldots, z_n)$, we have that $h_{jk} = H(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k})$ and $\omega = \frac{1}{2}\sum_{j,k} h_{jk} dz_j \wedge d\bar{z}_k$.

We define a hermitian structure on M to be Kahler if $d\omega = 0$. This implies that $h_{jk}(z) = \delta_{jk} + O(|z|^2)$.

3.1 Symplectic and Kähler Manifolds

A symplectic manifold is a pair (M, ω) where $d\omega = 0$ and $\omega^n \neq 0$, with ω a 2-form. We'll assume that M is compact. Then M being Kähler implies that M is symplectic, because $\int_M \omega^n \neq 0$, and so $0 \neq [\omega]^n \in H^{2n}(M)$, so each $[\omega^k]$ is nonzero.

Example 3.2. *Calabi and Eckmann proved that for $n, m \geq 1$, there was a complex structure on $S^{2n+1} \times S^{2m+1}$, and these can never be Kähler.*

In fact, any compact symplectic manifold has an almost complex structure.

Example 3.3. *$\mathbb{C}^n$ is Kähler with metric $\frac{i}{2}\sum_{i,j} \delta_{ij} dz_i \wedge d\bar{z}_j$.*

Example 3.4. *$\mathbb{P}^n$ has a Kähler structure, by taking on each U_j the sunftion $\rho_j([z]) = \frac{\sum |z_i|^2}{|z_j|^2} > 0$. On $U_j \cap U_k$, we have $|z_j|^2 \rho_j = |z_k^2| \rho_k$, we then take logs and apply $\partial\bar{\partial}$, and we find that $\partial\bar{\partial}\log(\rho_j) = \partial\bar{\partial}\log(\rho_k)$. So we set $\omega_j = -\frac{1}{2\pi i}\partial\bar{\partial}\log(\rho_j(z))$, and the metric we construct is the Fubini-Study metric.*

Let $N \subset M$. Then for all $p \in N$, there exist coordinates $(U, z_1, \ldots, z_n)$ on M around p Such that $N \cap U$ is described by $z_1 = \ldots = z_k = 0$. If M is Kähler and $N \subset M$ is a submanifold, then N is Kähler.

Thus, if M is a submanifold of $\mathbb{P}^n$, then M is Kähler. Thus, $[\omega] \in H^2(M) \cap H^2(M, \mathbb{Z})$ is necessary.

4 Lecture 4

Let's look at the real, smooth case. Let M be a compact oriented Riemannian manifold.

If V is a real vector space which is oriented with an inner product, then $\bigwedge^k(V^*)$ has an inner product as well.

Exercise 4.1. *Show that $\langle \alpha_1 \wedge \ldots \wedge \alpha_r, \beta_1 \wedge \ldots \wedge \beta_r \rangle = \det(\langle \alpha_i, \beta_j \rangle)$.*

Volume element $\Omega \in \bigwedge^n(V^*)$ given by $\xi_1 \wedge \ldots \wedge \xi_n$, where the ξ_i form an orthonormal basis.

We have a map $* : \bigwedge^k(V^*) \to \bigwedge^{n-k}(V^*)$, and $*(\xi_{i_1} \wedge \ldots \wedge \xi_{i_r}) = \operatorname{sign}(I, J)\xi_{j_1} \wedge \ldots \wedge \xi_{j_{n-r}}$. So then $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \Omega$.

Now, $*$ is an isomorphism, and it satisfies $*^2 = (-1)^{r(n-r)}$.

Back to the manifold M. We can define an inner product on forms α, β by $\int_M \alpha \wedge *\beta$. This is a positive definite bilinear form.

Now, we define $\delta = (-1)^{nr+1} * d*$, and it takes r-forms to $r-1$-forms, using this $*$ operator. We claim that d and δ are adjoints, that is, $(d\alpha, \beta) = (\alpha, \delta\beta)$.

$\int_M d\alpha \wedge *\beta = \int_M d(\alpha \wedge *\beta) - (-1)^r \alpha \wedge d*\beta$, but this will just be $-(-1)^r \int_M \alpha \wedge d * \beta$, up to sign, we can just insert a $*^2$ in front of d, and the signs work out.

Now, if $d\alpha = 0$, then $*\alpha$ may not be closed, but it is if and only if $\delta\alpha = 0$.

Proposition 4.2. *$d\alpha = \delta\alpha = 0$ if and only if $(d\delta + \delta d)\alpha = 0$.*

One direction is simple, for the other, we have $0 = \langle (d\delta + \delta d)\alpha, \alpha \rangle$, which using the adjoint property proves the result.

We call this operator Δ, the Laplace-Beltrami operator, or the Laplacian, and we call any form α with $\Delta\alpha = 0$ a harmonic form.

Exercise 4.3. $*\Delta = \Delta *$.

Why should we hope that every cohomology class has a harmonic form in it?

Heuristically, start with $d\alpha = 0$. Then $[\alpha]$ is the set of forms of the form $\alpha + d\beta$. Then $\|\alpha + td\beta\|^2 = \langle \alpha, \alpha \rangle + 2t\langle \alpha, d\beta \rangle + t^2 \|d\beta\|^2$. Now, suppose $\|\alpha\|^2$ is a maximum. Then for all β we'll have that $\langle \alpha, d\beta \rangle = 0$ and $\langle \delta\alpha, \beta \rangle = 0$, so we can assume that $\delta\alpha = 0$.

Theorem 4.4 (Hodge's Theorem)**.**

1. *$\mathscr{H}^k(M)$, the harmonic forms, is finite dimensional*
2. *$\mathcal{A}^r(M) = \mathscr{H}^k \oplus \Delta(\mathcal{A}^r(M))$.*

In particular, every form is a harmonic form, plus d of something plus δ of something. So then any form is $\alpha + d\beta + \delta\gamma$. But if it's closed, then $d\delta\gamma = 0$, which implies that $\delta\gamma = 0$, so for any closed form, it is of the form $\alpha = \eta + d\beta$.

Thus, $H^r_{dR}(M, \mathbb{R}) \cong \mathscr{H}^r(M)$.

Take a submanifold $Z^{n-k} \subset M^n$, then for any form in $H^{n-k}(M)$, we can restrict it to Z and integrate to get a map to $\mathbb{R}$. By Poincaré duality, this gives us a class $\eta_Z \in H^k(M)$.

Now, we define everything in a "hermitian way." So we take $\int_M \alpha \wedge *\bar{\beta}$, and note that $*$ takes (p, q) to $(n-q, n-p)$.

We define $\bar{\partial}^* = - * \partial *$ and $\partial^* = - * \bar{\partial} *$, and these are of type $(0, -1)$ and $(-1, 0)$ and match with ∂ and $\bar{\partial}$. So then we have $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, and we can write any form as a sum $\alpha = \alpha^{k,0} + \alpha^{k-1,1} + \ldots$.

And, we leave off with the fact that, on a Kähler manifold $\Delta_d = 2\Delta_{\bar{\partial}}$.

5 Lecture 5

We'll be working on a compact Kähler manifold. We know that $\Delta = 2\Delta_{\bar{\partial}}$.

Corollary 5.1. *If $\alpha \in A^k(M)$ is harmonic and $\alpha = \sum \alpha_{p,q}$, then $\Delta\alpha_{p,q} = 0$.*

Let $H^{p,q}(M)$ be the set of classes in $H^{p+q}(M)$ such that α has a representation in bidegree p, q.

Theorem 5.2. *$H^k_{dR}(M,\mathbb{C}) \cong \oplus_{p+q=k} H^{p,q}(M)$ and $H^{p,q}(M) \cong \mathscr{H}^{p,q}(M) = H^{p,q}_{\bar\partial}(M) \cong H^q(M,\Omega^p)$.*

We in fact have $H^{q,p}(M) = \overline{H^{p,q}(M)}$.

Now, if k is odd, we have $H^k(M) = H^{k,0} \oplus \ldots \oplus H^{0,k}$, and there is no middle term, so the dimension of $H^k(M)$ is even when k is odd.

The cup product will actually respect the bigrading: $H^{p,q} \cup H^{p',q'} \subset H^{p+p',q+q'}$. Voisin has found examples where every condition is satisfied except this property.

Why should all of these nice things be true? Let $\omega \in H^2(M,\mathbb{R}) \cap H^{1,1}(M)$. We have a Lefschetz map $L_\omega : A^* \to A^*$ by $\alpha \mapsto \omega \wedge \alpha$ increasing degree by $(1,1)$. We have $L_\omega^{n+1} = 0$. Now, we can define $Y : A^* \to A^*$ by $Y(\alpha) = (n-k)\alpha$ for $\alpha \in A^k$, and we have $[Y, L] = -2L$. Now, define $N_+ = (-1)^k * L *$ on A^*. This is the adjoint of L, and it satisfies $[Y, N_+] = 2N_+$ and $[N_+, L] = Y$, so this actually gives us a representation of the Lie algebra $\mathfrak{sl}_2$!

In this representation, L and N_+ are shifts, and the eigenvalues of Y are the degrees of forms, L increases degree and N_+ decreases. Then there's the Lefschetz Theorem which says, first, that $L^k_\omega : \bigwedge^{n-k}(T^*_p) \to \bigwedge^{n+k}(T^*_p)$ is an isomorphism. Moreover, it tells us that there are two types of cohomology classes $\bigwedge^{n-k}(T^*_p) = P^{n-k} \oplus L(\bigwedge^{n-k-2})$. We call this the Lefschetz decomposition for forms.

We in fact have the following relations:

1. $[\partial, L] = [\bar\partial, L] = [\partial^*, N_+] = [\bar\partial^*, N_+] = 0$
2. $[\bar\partial^*, L] = i\partial$, $[\partial^*, L] = -i\bar\partial$, $[\bar\partial, N_+] = i\partial^*$, $[\partial, N_+] = -i\bar\partial^*$.

These imply that $[\Delta_\partial, L] = [\Delta_\partial, Y] = [\Delta_\partial, N_+] = 0$, by just writing it out and putting L between the partials. This implies that the Lefschetz theorem holds if we replace forms by cohomology classes. We call this the Hard Lefschetz Theorem.

This gives another topological restriction: the betti numbers (even and odd separately) must be increasing to the middle degree.

Now, let us assume $\dim_{\mathbb{C}} M = 1$, so we are on a Riemann surface. So then $H^1 = H^{1,0} \oplus H^{0,1}$, and by the Dolbeault theorem, $H^{1,0} \cong H^{1,0}_{\bar\partial} \cong H^0(M,\Omega^1)$, the holomorphic 1-forms, that can be written $f(z)dz$ locally, for f holomorphic. Then there is $H^{0,0}$ and $H^{1,1}$, so the Hodge numbers are easy: $h^{0,0} = h^{1,1} = 1$ and $h^{1,0} = h^{0,1} = g$.

On a complex surface, things are slightly more interesting, and $H^{1,1}$ splits into $H^{1,1}_0 + \mathbb{C}\omega$, where subscript of zero will indicate the primitive cohomology.