Normal Functions

Brosnan

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The motivation is that Lefschetz used normal functions to prove the $(1,1)$ theorem, so maybe understanding normal functions will help prove the Hodge conjecture. The first person to explicitly define admissable normal functions seems to have been M. Saito in JAG and studied their zero loci and related them to the Hodge conjecture.

Theorem 1.1 (Theorem A). *Let $\bar{S}$ be a complex manifold and $\mathscr{H}$ a variation of pure Hodge structure of negative weight on S, a Zariski open subset of $\bar{S}$. Let $NF(S,\mathscr{H})^{ad}_{\bar{S}}$ be the group of admissable normal functions on S. For $\nu \in NF(S,\mathscr{H})^{ad}_{\bar{S}}$, set $Z(\nu)$ be the zero locus of ν. Then the closure of $Z(\nu)$ in $\bar{S}$ is a complex analytic subvariety of $\bar{S}$.*

Corollary 1.2. *If S is algebraic and $\bar{S}$ projective, then $Z(\nu)$ is an algebraic subvariety of S.*

Remarks: There are at least two proofs of the above. One by Brosnan and Pearlstein, using information from the mixed SL_2 orbit theorem of Kato, Nakayama and Usui. The other is by Schnell. He introduces an extension of the family $J(\mathscr{H}) \to S$ of Griffiths intermediate Jacobians. Uses that to extend the normal functions and the idea of Kato-Nakayama-Usui to compactify $J(\mathscr{H})$ is a more-or-less toroidal way.

There were some prior results by Saito in a JAG paper and by Brosnan-Pearlstein when $\dim S = 1$. Also when the singularity of ν vanishes.

There are two basics: the first is Hodge structures. The category of pure Hodge structures of weight $w \in \mathbb{Z}$ is by definition the category of pairs $(H, (H_{p,q})_{p,q\in\mathbb{Z}})$ such that H is finitely generated abelian group, $H^{p,q}$ are subspaces of $H_{\mathbb{C}}$ such that H is their direct sum and $\bar{H}^{pq} = H^{qp}$, and morphisms are maps preserivng the H^{pq}'s. We can replace finitely generated abelian group with A-modules for A a subgroup of $\mathbb{R}$, though $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are the only useful ones.

(defines MHS)

Theorem 1.3 (Deligne). *The category of MHS's is an abelian category, and if X is an algebraic variety over $\mathbb{C}$, then $H^k(X,\mathbb{Z})$ carries an MHS.*

The second part is much more difficult than the first, adn involves geometry and a lot of homological algebra. The first part is just a masterpiece of linear algebra. The idea is to define $I^{pq} = F^{[} \cap (\bar{F}^q \cap W_{p+q} + \bar{F}^{q-1} \cap W_{p+q-2} + \ldots)$, as in El Zein's notes. These give a bigrading, and a morphism of MHS's is a morphism that preserves the I^{pq}, and that's enough to show that it's an abelian category.

The category of pure Hodge structures is essentially semi-simple. If X is a smooth projective variety then $H^k(X)$ is a direct sum of irreducible Hodge structures. The category of MHS is not, there are nontrivial extension.

Exercise 1.4. *Take $\lambda \in \mathbb{C} \setminus \{0, 1\}$ and set $H_\lambda = H^1(\mathbb{P} \setminus \{0, \infty\}, \{1, \lambda\})$. Then H_λ is a nontrivial extension of $\mathrm{Gr}_2^W H_\lambda = \mathbb{Q}(-1)$ by $\mathrm{Gr}_0^W H_\lambda = \mathbb{Q}(0)$.*

In fact, we can explain the extension geometrically we have $0 \to H^1(\mathbb{P}^1, \{1, \lambda\}) \to H^1(\mathbb{P}^1 \setminus \{0, \infty\}, \{1, \lambda\}) \to H^1(\mathbb{P}^1 \setminus \{0, \infty\}) \to 0$ which is naturally $0 \to \mathbb{Z}(0) \to H_\lambda \to \mathbb{Z}_{-1} \to 0$.

To see that the extension is nontrivial, really need to calculate it, but to calculate it, we need to find where it goes.

Theorem 1.5 (Carlson)**.** *Let H, K be pure Hodge structures with K of lower weight than H and $H_{\mathbb{Z}}$ torsion free. Then $\mathrm{Ext}^1_{MHS}(H, K) = \underline{\mathrm{hom}}(H, K)/F^0 \,\mathrm{hom}(H, K) + \mathrm{hom}(H_{\mathbb{Z}}, K_{\mathbb{Z}})$.*

There, $\underline{\mathrm{hom}}(H, K)$ is a pure Hodge structure of weight $k-h$ where $F^p\underline{\mathrm{hom}}(H, K)$ are the homomorphisms $\phi(F^k) \subset F^{k+p}$.

Proof. We'll define a map by taking an extension $0 \to K \to E \xrightarrow{\pi} H \to 0$. We can always find $\phi_{\mathbb{Z}} : H_{\mathbb{Z}} \to E_{\mathbb{Z}}$ of finitely generated abelian groups splitting the extension. On the other hand, maps $\pi : E \to H$ of mixed Hodge structure are strict with respect to the Hodge filtration, so $\pi(F^p_E) = F^p H \cap \pi(E)$, and from this it follows that we can find a map $\phi : H_{\mathbb{C}} \to E_{\mathbb{C}}$ such that ϕ_F is in the right place in the filtration. □

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We'll finish proving the theorem of Carlson.

Theorem 2.1.

$$\mathrm{Ext}^1_{MHS}(H, K) = \mathrm{hom}(H_{\mathbb{C}}, K_{\mathbb{C}})/F^0\underline{\mathrm{hom}}(H, K) + \mathrm{hom}(H_{\mathbb{Z}}, K_{\mathbb{Z}})$$

Sketch continued:

So if we start with $0 \to K \to E \xrightarrow{\pi} H \to 0$, because $H_{\mathbb{Z}}$ is torsion free, we can find a splitting $\sigma_{\mathbb{Z}} : H_{\mathbb{Z}} \to E_{\mathbb{Z}}$ because $\pi : E \to H$ is strict with respect to F, we can find a splitting $\sigma_F : H_{\mathbb{C}} \to E_{\mathbb{C}}$ preserving the Hodge filtration.

Using the fact that $F^p = \oplus_{p' \geq p} I^{p',q}$to decude that π is strict with respect to teh Hodge filtration. Define $cl(E) = \sigma_F - \sigma_{\mathbb{Z}} : H_{\mathbb{C}} \to K_{\mathbb{C}}$. But $cl(E)$ depends on some choices: we could add any morphism $\phi_{\mathbb{C}} \in F^0(\mathrm{hom}(H_{\mathbb{C}}, K_{\mathbb{C}}))$ to σ_F, and we could add any morphism $\phi_{\mathbb{Z}} \in \hom(H_{\mathbb{Z}}, K_{\mathbb{Z}})$ to $\sigma_{\mathbb{Z}}$. Modding out by these choices determines the morphism we need.

The second step is to product a map backwards. Suppose that $f \in \hom(H_{\mathbb{C}}, K_{\mathbb{C}})$. Define $E_0 = H \oplus K$, and define $T_f \in \mathrm{End} H_{\mathbb{C}} \oplus K_{\mathbb{C}}$ by $(h,k) \mapsto (h, k + f(h))$. Define an extension E_f by taking the underlying group to be E_0, $W_n E_f = W_n E_0$ and $F^* E_f = T_f(F^*)$. Then T_f induces the identity on Gr_0^W and therefore $F^* E_f = F^* E_0$ on Gr_2^W and therefore E_f is a Hodge structure. We need to check that $cl(E_f) = f$.

Most important case of the theorem is if $H = \mathbb{Z}$ and K is of negative weight, we set $J(K) = \mathrm{Ext}_{MHS}(\mathbb{Z}, K) = K_{\mathbb{C}}/F^0 K_{\mathbb{C}} + K_{\mathbb{Z}}$, this is a complex torus. To check this we need to check that $K_{\mathbb{Z}}$ is discrete in $F_{\mathbb{C}}/F^0$. In fact, $\mathrm{Ext}^1_{MHS}(K,H) = \mathrm{Ext}^1_{MHS}(\mathbb{Z}(0), \hom(H,K))$.

In the exercise, we have $0 \to \mathbb{Z}(0) \to H_\lambda \to \mathbb{Z}(-1) \to 0$. In that case, we have $\mathrm{Ext}^1_{MHS}(\mathbb{Z}(-1), \mathbb{Z}(0)) = \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-1) \otimes \mathbb{Z}(1), \mathbb{Z}(0) \otimes \mathbb{Z}(1))$ (all previous $\mathbb{Z}$'s are $\mathbb{Z}(0)$)

So this is just $\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}(1))$, which is $\mathbb{C}/\mathbb{Z}(1) = \mathbb{C}/2\pi i\mathbb{Z}$ and by the exponential map, this is $\mathbb{C}^*$.

In general, $J(H)$ is called the Griffiths intermediate Jacobian, and is generally not algebraic. However, when $H_{\mathbb{C}} = H^{-1,0} \oplus H^{0,-1}$ is polarized, then $J(H)$ is an abelian variety: $J(H_1(C))$ is the Jacobian of the curve.

If the weight is -1, then $J(H)$ is a compact complex group. The reason is that we need to show that $H_{\mathbb{Z}}$ i sdiscrete. Look at $H_{\mathbb{R}} \to H_{\mathbb{C}}/F^0$. Ths is injective because $H_{\mathbb{R}} \cap F^0$ is zero, both have the same dimension as real vector spaces, and so it is an isomorphism of real vector spaces. Since $H_{\mathbb{Z}}$ is discrete in $H_{\mathbb{R}}$, it is in $H_{\mathbb{C}}/F^0$ as well.

Example 2.2. *Let C be a smooth projective complex curve, and $D = \sum_{p \in |D|} n_p[p]$ a divisor of degree 0. We get a long exact sequence of MHS as follows:*

$$0 = H^1(C, C \setminus D) \to H^1(C) \to H^1(C \setminus D) \to H^2(C, C - D) \to H^2(C)$$

and the last two terms are $\oplus_{p \in D} \mathbb{Z}(-1)$ and $\mathbb{Z}(-1)$. The divisor D then gives a map $\mathbb{Z}(-1) \to \mathbb{Z}(-1)^D$ by sending $\frac{1}{2\pi i}$ to $\frac{1}{2\pi i} \sum n_p$.

So D gives an extension $[E_d] \in \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-1), H^1(C)) = \mathrm{Ext}_{MHS}(\mathbb{Z}(0), H^1(C)(1)) = J(H^1(C))$, which is just the Jacobian of C, and the map is the Abel-Jacobi map $\mathrm{AJ} : \mathrm{Div}^0(C) \to \mathrm{Jac}(C)$.

2.1 Normal Functions

Let D be a divisor on a surface $\sum n_i D_i$ where the D_i are irreducible curves not contained in fibers of a map $f : X \to \mathbb{P}^1$.

For each $p \in \mathbb{P}^1$ we get a curve as the fiber, and on a dense open $U \subset \mathbb{P}^1$, these curves are smooth.

Assume that for one and helce all $p \in U$, we have $D_p = D \cdot X_p$ has degree 0. Then the above construction applied to the curves X_p gives a section of the family whose fibers are the Jacobians of X_p. For eahc point, we've got the Abel-Jacobi map, and this section is called a normal function.

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Last time, we looked at $\mathrm{Ext}^1_{MHS}(\mathbb{Z}, H) = J(H) = H_{\mathbb{C}}/F^0H + H_{\mathbb{Z}}$ is the Griffiths intermediate Jacobian for H pure of negative weight, by Carlson's formula.

For X a smooth projective variety, $J^p(X) = J(H^{2p-1}(X, \mathbb{Z})(-p))$ which has weight -1.

Recall from Cattani that a variation of Hodge structure of weight k on a complex manifold S consists of a pair (H, F) where H is a local system on S, F^* is a decreasing filtration on $H \otimes_{\mathbb{Z}} \mathscr{O}_S$ and at every point $s \in S$, we have $(H, F^p)_s$ defines a Hodge structure, along with Griffiths Transversality, that $\nabla F^p \subset F^{p-1} \otimes \Omega_S$.

Now, Griffiths notices that if $f : X \to S$ is a smooth projective morphism, then if you set $\mathscr{H} = R^k f_* \mathbb{Z}$ so that $\mathscr{H}_s = H^k(X_s, \mathbb{Z})$, then we have a variation of Hodge structures.

A variation of MHS is a triple (V, F, W), with V a local system, F and W appropriate filtrations, see Cattani.

And, just to be clear:

Definition 3.1 (Local System)**.** *A local system of A modules on S for A a ring is a locally constant sheaf of A-modules. If S is connected, then local systems of A modules are in bijection with $\pi_1(S, s) - A$ modules.*

Definition 3.2 (Normal functions)**.** *Let S be a complex manifold, $\mathscr{H}$ a variation of pure Hodge structures on S. The group of normal functions on S is the group $NF(S, \mathscr{H}) = \mathrm{Ext}^1_{VMHS(S)}(\mathbb{Z}, \mathscr{H})$.*

In this, $VMHS(S)$ denotes teh abelian category of variations of MHS on S and $\mathbb{Z}$ denotes the constant pure Hodge structure on S which is $\mathbb{Z}(0)$ at every point, and the weight of H is negative.

Given $\mathscr{H}$, we have a family $J(\mathscr{H}) \to S$ of intermediate Jacobians and the fiber at $s \in S$ is $J(H_s)$, so by restriction, an element $\nu \in NF(S, \mathscr{H})$ determines a section σ_ν of $J(\mathscr{H})/S$.

Fact: σ_ν determines γ. In other words, there is an injection $NF(S, \mathscr{H})$ to the group of section $\sigma : S \to J(\mathscr{H})$ of a family $J(\mathscr{H}) \to S$.

The idea of hte proof is to try to use the section $\sigma : S \to J(\mathscr{H})$ to construct an extension of mixed Hodge structures. What would be nice would be to have a tautological extension of $\mathbb{Z}$ by $\mathscr{H}$ sitting over $J(\mathscr{H})$. Then you can pull it back by σ and can ignore Griffiths transversality and find such a tautological extension to get injectivity.

Example 3.3. *Look at $S = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ and $\mathscr{H} = \mathbb{Z}(1)$, then $\mathrm{Ext}^1_{VMHS(S)}(\mathbb{Z}, \mathbb{Z}(1)) = \mathscr{O}^*_S(S)$, and we recall from last time that we constructed an element corresponding to $\lambda \in \mathscr{O}^*_S(S)$, and we gave it as an extension $0 \to \mathbb{Z} \to H^1(\mathbb{P}^1 \setminus \{0, \infty\}, \{1, \lambda\}) \to \mathbb{Z}(-1) \to 0$, and this is in $\mathrm{Ext}(\mathbb{Z}(-1), \mathbb{Z})$, which is isomorphic to $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}(1))$.*

3.1 Singularities

Suppose that $j : S \to \bar{S}$ open immersion of complex manifolds, and $\mathscr{H}$ a VHS on S. Typical situation: $f : X \to \bar{S}$ with $X, \bar{S}$ smooth projective varieties, and by generic smoothness, there is a dense oepn $S \subset \bar{S}$ where f is smooth. Therefore, if we set $\mathscr{H} = R^k f_* \mathbb{Z}$, we get that $\mathscr{H}$ is a variation of Hodge structure on S, but won't in general extend to $\bar{S}$.

Take $\nu \in NF(S, \mathscr{H})$ then $cl(\nu) \in \mathrm{Ext}_{Sheaves(S)}(\mathbb{Z}, \mathscr{H}_{\mathbb{Z}}) = H^1(S, \mathscr{H}_{\mathbb{Z}})$ is a topological invariation of ν, the singularity of ν at a point $s \in \bar{S} \backslash S$. We define it to be $sing_s(\nu)$ which captures the topology of ν near s as follows: take a small ball around s in $\bar{S}$. Defien $sing_s(\nu)$ to be the image of $cl(\nu)$ under the map $H^1(S, \mathscr{H}_{\mathbb{Z}}) \to H^1(B \cap S, \mathscr{H}_{\mathbb{Z}})$. As long as B is small enough, this is independent of the ball.

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4.1 Hodge Classes and Normal Functions

Lefschetz realized that you can start with a Hodge class $\alpha \in H^2(X)$ where $X \to \mathbb{P}^1$ is a smooth projective algebraic surface, and if you assume that α is primitive, meaning that $\alpha|_{X_s} = 0$ for $s \in \mathbb{P}^1$ such that X_s is smooth, then there is a normal function associated to α in $NF(U, \mathscr{H}$ where U is teh smooth locus of $\mathbb{P}^1$.

Definition 4.1. *Let X be a smooth projective complex variety. Then Deligne sets $\mathbb{Z}(p)_D = \mathbb{Z}(p) \to \mathscr{O}_X \to \Omega^1_X \to \ldots \to \Omega^{p-1}_X$ in degrees 0 to p.*

Note that $\mathbb{Z}(1)_D$ is just $\mathbb{Z}(1) \to \mathscr{O}_X$, and so gives the exponential sequence.

Thus, $\mathbb{Z}(1)_D \cong \mathscr{O}^*_X[-1]$, and $H^2(X, \mathbb{Z}(1)_D) = H^0(X, \mathbb{Z}(1)_D[2]) = H^0(X, \mathscr{O}^*_X[1]) = H^1(X, \mathscr{O}^*_X) \cong \mathrm{Pic}\, X$.

Definition 4.2. *For $p \in \mathbb{Z}$, $Hodge^{2p}(X) = H^{2p}(X, \mathbb{Z}(p)) \cap H^{p,p}$*

Proposition 4.3. *If X is a smooth projective variety, then is an exact sequence $0 \to J(H^{2p-1}(X)(p)) \to H^{2p}(X, \mathbb{Z}(p)_D) \xrightarrow{cl} Hodge^{2p}(X) \to 0$.*

Strictly, we've been working with hypercohomology, but we're not going to worry about distinguishing it.

Suppose that $f : X \to \bar{S}$ is a morphism with $X, \bar{S}$ smooth projective and irreducible, and S the smooth locus. Write $Prim(X/S) = \{\alpha \in Hodge^{2p}(X) : \alpha|_{X_s} = 0 \forall s \in s\}$ to be the primary classes.

Given $\alpha \in Prim^p(X/S)$, we can find a lift $\tilde{\alpha}$ of α ot $H^{2p}(X, \mathbb{Z}(p)_D)$. Then for every $s \in S$ we have $cl(\tilde{\alpha}_s) = \alpha|_s = 0 \in H^{2p}, X_s, \mathbb{Z}(p)_D)$.

So for every $\tilde{\alpha}$, we get a map $s \mapsto \tilde{\alpha}_s \in J(H^{2p-1}(X_0)(-p))$ and so we write $\mathscr{H} = R^{2p-1} f_* \mathbb{Z}(p)|_S$.

Proposition 4.4. *The association $s \mapsto \tilde{\alpha}_s$ defines a normal function $\gamma(\tilde{\alpha}) \in NF(S, \mathscr{H})$.*

The dependence on $\tilde{\alpha}$ is pretty simple: any other lift of α will differ from $\tilde{\alpha}$ by an element of $J(H^{2p-1}(X)(p))$.

Proposition 4.5. *We have a map* $Prim^p(X/S) \to NF(S, \mathscr{H})/J(H^{2p-1}(X)(-p))$.

Proposition 4.6. $0 \to J(H^{2p-1}(X)(-p)) \to H^{2p}(X, \mathbb{Z}(p)_D) \xrightarrow{cl} Hodge^{2p}(X) \to 0$.

So we have a triangle $\Omega_X^{\leq p-1}[-1] \to \mathbb{Z}(p)_D \to \mathbb{Z}(p) \to \Omega_X^{\leq p-1}$ and so we jsut need to know $\mathbb{H}(X, \Omega_X^{\leq p-1})$. In fact, we have $\Omega^{\geq p} \to \Omega^* \to |\Omega^{\leq p-1} \to \Omega^{\geq p}[1]$, and by degeneration of Hodge to deRham spectral sequence, we know that $H^n(X, \Omega_X^{\geq p}) = F^p H^n(X, \mathbb{C})$.

So $H^n(X, \Omega_X^*) = H^n(X, \mathbb{C})$, and therefor $H^n(X, \Omega^{\leq p-1}) = H^n(X, \mathbb{C})/F^p H^n(X/\mathbb{C})$.

So we have $H^{2p-1}(X, \mathbb{Z}(p)) \to H^{2p-1}(X, \mathbb{C})/F^p \to H^{2p}(X, \mathbb{Z}(p)_D) \to H^{2p}(X, \mathbb{Z}(p)) \to H^{2p}(X, \mathbb{C})/F^p$.

4.2 A construction of Green-Griffiths

If X is a smooth projective variety with $\dim X = 2n$ and $\mathscr{O}(1)$ fixed very ample line bundle, from this let's construct a sequence of normal functions on a sequence of spaces.

First, we construct the spaces. For each $d \in \mathbb{Z}_{>0}$, set $\mathbb{P}_d = \mathbb{P}(H^0(X, \mathscr{O}_X(d)) = |\mathscr{O}_X(d)|$ is the complete linear system on $\mathscr{O}_X(d)$.

Now, write $Prim(X) = \{\alpha \in Hodge^{2n} X : \alpha|_D = 0 \text{ for } D \text{ a smooth divisor in } \mathscr{O}_X(d)\}$. Now for each d we have an incidence variety $\mathcal{X}_d = \{(x, f) \in X \times \mathbb{P}_d : f(x) = 0\}$ and this is smooth over some dense open $U \subset \mathbb{P}_d$. Set $\mathscr{H}_d = R^{2n-1}\pi_{d*}\mathbb{Z}(p)$.

By definition, if $\alpha \in Prim(X)$, and $\pi : \mathcal{X}_d \to X$ the projection, then $\pi^*\alpha \in Prim^{2n}(\mathcal{X}_d/\mathbb{P}_d)$, and so we get a map $Prim(X) \to NF(U_d, \mathscr{H})/J(H^{2n-1}\mathcal{X}(n))$.

Now let $s \in \mathbb{P}_d \setminus U_d$, the set of singular hyperplane sections. Then Green-Griffiths noticed that teh singular $sing_s\nu$ of the normal function ν associated to a primitive class $\alpha \in Prim(X)$ is related to the restriction of α to the divisor $X_s \subset \mathcal{X}_d$

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Theorem 5.1 (R. Thomas)**.** *The following are equivalent:*

1. *The Hodge Conjecture holds for all smooth projective varieties X*
2. *For all smooth projective varieties $X \subset \mathbb{P}^M$, of even dimension $2n$ and all Hodge classes $\alpha \in H^{2m}(X, \mathbb{Z}(n))$ there exist a $k \in \mathbb{Z}_{>0}$ and a $D \in |\mathscr{O}(k)|$ such that $0 \neq \alpha|_D \in H^{2n}(D, \mathbb{Z}(n))$.*

There is a similar result in Saito's "Admissable Normal Functions"

Proof. $\Rightarrow$: If HC holds, then given a Hodge class $\alpha \in H^{2n}(X, \mathbb{Z}(n))$, there exists another Hodge class which is algebraic $\beta = [Z]$ such that $\alpha \cup \beta \neq 0$. "Clearly" $Z \subset D$ for some divisor D, therefore $\alpha|_D \neq 0$. ☐

Theorem 5.2 (Theorem B). *Suppose that X is even dimensional as above and $\alpha \in PrimX$. Then for $k >> 0$, we have $sing_s \nu(s) = 0$ if and only if $\alpha|_{\mathcal{X}_s} = 0$, where $\mathcal{X}$ is the universal hyperplane over $\mathbb{P}$.*

This is a result of Brosnan, Fong, Nie, and Pearlstein, and of de Cataldo and Migliorini, and uses the decomposition theorem for perverse sheaves.

ν an admissable normal function implies that $sing_s \nu \in IH^*(B \cap S, \mathscr{H}) \subseteq H^*(B \cap S, \mathscr{H})$ for B a small contractible ball.

Construction: let Δ be the unit disc, and Δ^* the punctured disc. Suppose that $\mathscr{V}$ is a VMHS on $(\Delta^*)^r$. Pick $s_0 \in (\Delta^*)^r$. Then the underlying local system $\mathscr{V}_{\mathbb{Z}}$ is a $\mathbb{Z}^r = \pi_1((\Delta^*)^r, s_0)$-module determined by the action of γ commuting invertible operators $T_1, \ldots, T_r$ on $\mathscr{V}_{\mathbb{Z}}$.

Theorem 5.3 (Borel). *The T_i are quasi-unipotent operators. There exists positive integers a, b such that $(T^a - 1)^b = 0$.*

Remark: T_i are quasi-unipotent on $\mathrm{Gr}^W \mathscr{V}$ implies that they are on $\mathscr{V}$, so the monodromy theorem holds for MHS.

If $\mathscr{V}$ is polarizable variation, then we can find $\Delta^r \to \Delta^r$ such that the monodromy is unipotent.

5.1 Monodromy Filtration

Let V be a vector space and N a nilpotent operator, $\log T$.

Theorem 5.4. *There exists a unique increasing filtration W on V satisfying $N(W_*) = W_{*-2}$, $N^a Gr^W_a(V) = \mathrm{Gr}^W_{-a}(V)$ for $a > 0$.*

Relative Weight Filtration: Let N be nilpotent operator on a finite dimensional vector space V equipped with an increasing filtration W_*.

Theorem 5.5. *there exists at most 1 increasing filtration $M = M(N, W)$ of V satisfying $NM_* \subset M_{*-2}$ and $N^a \mathrm{Gr}^M_{k+a} \mathrm{Gr}^W_k V \to \mathrm{Gr}^M_{k-a} \mathrm{Gr}^W_k V$ an isomorphism for all k and all $a \geq 0$.*

Deligne noticed in Weil 2 that ℓ-adic sheaves coming from geometry always have this property.

Definition 5.6 (Admissable). *Suppose $\mathscr{V} \in VMHS(\Delta^*)$ with unipotent monodromy. We say $\mathscr{V}$ is admissible relative to Δ if:*

1. $\mathrm{Gr}^W_k \mathscr{V}$ *is polarizable*
2. *The Hodge filtration extends to holomorphic subbundles of $\mathscr{V}_{can}$ =Deligne canonical extension of $\mathscr{V} \otimes_{\mathbb{Z}} \mathscr{O}_{\Delta^*}$ to a vector bundle on Δ such that the connection has regular singular points.*
3. *If we pick $s_0 \in \Delta^*$, and use $V = \mathscr{V}_{s_0}$ and N the monodromy, then $M(N, V)$ exists.*

Define $VMHS(\Delta^*)^{ad}_{\Delta}$ the category of variations which are admissible after pullback to make the monodromy unipotent.

Def (Kashiwara "A study of variations of mixed Hodge structure") suppose that $\bar{S}$ a complex manifold, $Z \subset \bar{S}$ a closed algebraic set with complement S. Then $\mathscr{V}$ a VMHS on S is admissible relative to $\bar{S}$ is whenever we map $\Delta^* \to S$ such that we can complete to $f : \Delta \to \bar{S}$, we have $f^*\mathscr{V}$ admissible relative to Δ.

Then we set $VMHS(S)^{ad}_{\bar{S}}$ the abelian category of admissible variations.

If S is quasi-projective, then $VMHS(S)^{ad}_{\bar{S}} = VMHS(S)^{ad}_{\bar{S}'}$ for any projective completion of S.

Define (Saito) $NF(S, \mathscr{H})^{ad}_{\bar{S}} = \mathrm{Ext}^1_{VMHS(S)^{ad}_{\bar{S}}}(\mathbb{Z}, \mathscr{H})$ for $\mathscr{H}$ a variation of pure Hodge structure on S.

Note: polarizable VHS are always admissible.

Recall the big claim that if ν is an admissible normal funciton, then $Z(\nu)$ is algebraic.

We observe that $NF(S, \mathbb{Z}(1)) = \mathrm{Ext}^1_{VMHS}(\mathbb{Z}, \mathbb{Z}(1)) = \mathscr{O}^*_{S^{an}}$, so we need an extension where the extension class is e^z to see why we really need admissible.

As an example, take $S = \mathbb{P}^1 \setminus \{0\} = \mathbb{A}^1$. We define an extension $0 \to \mathbb{Z}(1) \to E \to \mathbb{Z}(0) \to 0$ which violates the algebraicity of the zero locus.

Take $E_{\mathbb{Z}} = \mathbb{Z}(2\pi i)e_{-2} + \mathbb{Z}e_0$. Checking carefully, the extension class is e^z, because the Hodge filtration doesn't extend to a subbundle!

Exercise 5.7. *Check that the problem is with extension of F^*, the Hodge filtration.*

Example 5.8. *Let $\mathscr{V}_{\mathscr{O}} = \mathscr{O}_S a \oplus \mathscr{O}_S b \oplus \mathscr{O}_S c$ and $\mathscr{V}_{\mathbb{Z}}$ generated by $a + \frac{\log s}{2\pi i} c$, b and c. Then take $F^0 = \mathscr{O}a + \mathscr{O}(b + ic)$ and $W_{-1} = \mathscr{O}b + \mathscr{O}c$, $W_0 =$everything and $W_{-2} = 0$.*

Exercise 5.9. *Show that relaive weight filtration chosen exists here.*

Point: $NV \cap W_{-1} \subset NW_{-1}$ and the problem in this case is the weight filtration.