Homework 7 Solutions

1 a Look at the left hand side of the equation. It gives $\dim V_2 - \dim V_1 - \dim V_3$. We want to conclude that this is zero. We note that we have $f_1 : V_1 \to V_2$ and $f_2 : V_2 \to V_3$, with f_1 injective and f_2 surjectve. The rank-nullity theorem says that $\dim V_1 = \dim\ker f_1 + \dim \mathrm{im} f_1$, and that $\dim V_2 = \dim\ker f_2 + \dim \mathrm{im} f_2$. Now, using injectivity and surjectivity, we get $\dim V_1 = \dim \mathrm{im} f_1$ and $\dim V_2 = \dim\ker f_2 + \dim V_3$. Substituting back, we obtain $\dim\ker f_2 + \dim V_3 - \dim \mathrm{im} f_1 - \dim V_3 = \dim\ker f_2 - \dim \mathrm{im} f_1$. But by the hypothesis of the problem, this is zero, as desired.

b Fix some n. Then we have $\sum_{i=1}^{n}(-1)^i \dim V_i$. We can write this as $-\dim V_1 + \sum_{i=2}^{n-1} \dim V_i + (-1)^n \dim V_n$. The Rank-Nullity theorem says that for V_i with $i \neq n$, we have $\dim V_i = \dim\ker f_i + \dim \mathrm{im} f_i$, and that $\dim\ker f_1 = 0$. Thus, this becomes $\dim \mathrm{im} f_1 + \sum_{i=2}^{n-1}(-1)^i(\dim\ker f_i + \dim \mathrm{im} f_i) + (-1)^n \dim V_n$. By the hypothesis, the middle part is a telescoping sum, and so this becomes $\dim \mathrm{im} f_1 - \dim\ker f_2 + (-1)^{n-1} \dim V_n + (-1)^n \dim V_n$. The hypothesis for the problem causes the first two to cancel, and the last two just have opposite signs, and so cancel. Thus, we obtain $\sum_{i=1}^{n}(-1)^i \dim V_i = 0$.

3 a Let M be a finitely generated R module. Then by problem 8c on Homework 5, $\hom_R(M, N)$ is finitely generated if M and N are. Note that $M^* = \hom_R(M, R)$. By hypothesis, M is finitely generated, and R is generated by 1_R, and so is finitely generated. Thus, M^* is finitely generated.

b So now we assume that M is finitely generated and free. This means that it has a linearly independent generating set $m_1, \ldots, m_n$. To see that M^* is also free, we must merely show that it also has one. Define $m_i^* : M \to R$ to be the function which takes $\sum_{i=1}^{n} a_i m_i$ to a_i. We have three things to show: that the m_i are homomorphisms, that they are linearly independent, and that they are spanning.

To see that they are homomorphisms, we let $m, n \in M$ and $r, s \in R$ arbitrary with $m = \sum a_i m_i$ and $n = \sum b_i m_i$. Then $rm + sn = \sum r a_i m_i + s b_i m_i = \sum (r a_i + s b_i) m_i$. So $m_i^*(rm + sn) = r a_i + s b_i$. But also, $r m_i^*(m) + s m_i^*(n) = r a_i + s b_i$, and so the m_i^* are homomorphisms.

For linear independence, let $f = \sum a_i m_i^* = 0$, and we need to show that $a_i = 0$ for all i. For each j, we have $f(m_j) = \sum a_i m_i^*(m_j) = \sum a_i \delta_{ij} = a_j = 0$. Thus, for each j, $a_j = 0$, so we have linear independence.

To see that it is a spanning set, let $f \in M^*$ arbitrary. Set $a_i = f(m_i)$. Claim: $f = \sum a_i m_i^*$. To justify this, take $m \in M$ arbitrary. We want to show that $\sum a_i m_i^*(m) = f(m)$. Now, we can write $m = \sum b_i m_i$. So then, as f is linear, $f(m) = f(\sum b_i m_i) = \sum b_i f(m_i) = \sum b_i a_i$. However, also, $m_i^*(m) = b_i$, and so $\sum a_i m_i^*(m) = \sum a_i b_i$. The two are equal, and so $f = \sum a_i m_i^*$, as desired.

4 a So see that $\phi : M \to (M^*)^*$ is a homomorphism of R-modules, we need that for all $m, n \in M$ and $r, s \in R$, $\phi(rm+sn) = r\phi(m)+s\phi(n)$. So let $m, n \in M$ and $r, s \in R$ be arbitrary. We need to show that ϕ_{rm+sn} and $r\phi_m + s\phi_n$ are equal. These are both maps $M^* \to R$, so it suffices to show that they are equal on any homomorphism $f : M \to R$. So fix $f : M \to R$ arbitrary. Then $\phi_{rm+sn}(f) = f(rm + sn) = rf(m) + sf(n) = r\phi_m(f) + s\phi_n(f)$, as desired, and so we have a homomorphism.

b False. Let $R = \mathbb{Z}/4\mathbb{Z}$ and $M = \mathbb{Z}/2\mathbb{Z}$. Then M^* and $(M^*)^*$ are also isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and the map ϕ is an ismorphism. However, M is not a free R-module, because for all $x \in M$, $2x = 0$, with $2 \neq 0$ in R.

6 Denote by ϕ the map $\hom(L, M) \times \hom(M, N) \to \hom(L, N)$ by $(f, g) \mapsto g \circ f$. Now, fix $f \in \hom(L, M)$ and let $g_1, g_2 \in \hom(M, N)$, $r, s \in R$ arbitrary. To show that ϕ is linear in g, we must show that $\phi(f, rg_1 + sg_n) = r\phi(f, g_1)+s\phi(f, g_2)$. Starting from the first, we have $(rg_1+sg_2)\circ f$. Now this is defined to be $rg_1 \circ f + sg_2 \circ f$, which is $r(g_1 \circ f) + s(g_2 \circ f) = r\phi(f, g_1)+s\phi(f, g_2)$. Similarly, if $f_1, f_2 \in \hom(L, M)$ and $g \in \hom(M, N)$, we must show that $\phi(rf_1 + sf_2, g) = r\phi(f_1, g) + s\phi(f_2, g)$, which proceeds by the same argument.

7 a Let $y = (y_1, \dots, y_m)$ and $z = (z_1, \dots, z_m)$ be two solutions and let $r \in R$ arbitrary. We must show that ry and $y - z$ are solutions. The equations are $\sum_j a_{ij}x_j = 0$, one for each i. Now, we plug in ry, and obtain $\sum_j a_{ij}ry_j = r\sum_j a_{ij}y_j = r0 = 0$, and so ry is also a solution. Now we try $y-z$, and obtain $\sum_j a_{ij}(y_j - z_j) = \sum_j a_{ij}y_j - \sum_j a_{ij}z_j = 0 - 0 = 0$. Thus, we have a submodule.

b Let $x = (x_1, \dots, x_m)$. Then, $f(x) = (\sum_j a_{1j}x_j, \dots, \sum_j a_{nj}x_j)$. So, let $y = (y_1, \dots, y_m)$ be in S. Then $f(y) = (0, \dots, 0)$, and so $y \in \ker f$. Similarly, if $y \in \ker f$, then $f(x) = 0$, and so we have $\sum_j a_{ij}y_j = 0$ for all i, and so y is a solution to the system of equations, and so lies in S. Thus, $S = \ker f$.