Homework 3 Solutions

1 (a) Assume that G is abelian. Then let $(x, y), (x', y') \in G$. Then $(x, y)(x', y') = (x', y')(x, y)$. Using the group law, this is $(xx', yy') = (x'x, y'x)$, which means that $xx' = x'x$ and $yy' = y'y$, and so G_1, G_2 are both abelian. Conversely, assume that G_1, G_2 are abelian. Let $(x, y), (x', y') \in G$. Then $(x, y)(x', y') = (xx', yy')$. As G_1, G_2 are abelian, this is just $(x'x, y'y) = (x', y')(x, y)$, and so G is abelian.

(b) This is false, as we can take $G_1 = \mathbb{Z}/2\mathbb{Z}$ and $G_2 = \mathbb{Z}/3\mathbb{Z}$. Then $G \cong \mathbb{Z}/6\mathbb{Z}$, and G has normal subgroup $\{0, 3\}$, and so isn't simple.

4 This is false, as if R_1, R_2 are skew fields, look at the elements of $R = R_1 \times R_2$ given by $(1_{R_1}, 0_{R_2})$ and $(0_{R_1}, 1_{R_2})$. Neither of these is zero in R. However, their product is $(0_{R_1}, 0_{R_2}) = 0_R$, and so they are zero divisors. This implies that they cannot be units, and so not every element of R is invertible.

5 (a) For this whole part of the problem, let $f, g, h \in \mathcal{F}(X, G)$ and $x \in X$ be arbitrary.

Closure: By definition, $(f * g)(x) = f(x)g(x)$. As G is a group, $f(x)g(x)$ is an element of G. Thus, $f * g$ defines a function by $x \mapsto f(x)g(x)$ which takes elements of X to elements of G, and so $f * g \in \mathcal{F}(X, G)$.

Associativity: We start with $(f*g)*h(x)$. This is equal to $(f(x)g(x))h(x)$, by the definition of $*$. Now, as G is associative, this is equal to $f(x)(g(x)h(x))$. By the definition of $*$ again, we can write this as $(f * (g * h))(x)$, and so $(f * g) * h = f * (g * h)$.

Identity: Define the function $e : X \to G$ by taking every $x \in X$ to e_G, the identity in G. Now, we look at $(f * e)(x) = f(x)e(x) = f(x)e_G = f(x) = e_G f(x) = e(x)f(x) = (e * f)(x)$. Thus, e is the identity for the operation $*$.

Inverses: Define the function f_{inv} by for all $x \in X$, $f_{inv}(x) = f(x)^{-1}$. We claim that this is the inverse function for f under $*$. We have $(f_{inv} * f)(x) = f_{inv}(x)f(x) = f(x)^{-1}f(x) = e_G = f(x)f(x)^{-1} = f(x)f_{inv}(x) = (f * f_{inv})(x)$, which shows that this is the case.

Commutativity: Assume that G is an abelian group. Then $(f * g)(x) = f(x)g(x) = g(x)f(x) = (g * f)(x)$, and so $*$ is commutative. Now assume that $*$ is commutative. For $a \in G$, we define $f_a \in$

$\mathcal{F}(X,G)$ to be the function taking all $x \in X$ to $a \in G$. Let $a,b \in G$. Then $ab = f_a(x)f_b(x) = (f_a * f_b)(x) = (f_b * f_a)(x) = f_b(x)f_a(x) = ba$, and so G is abelian.

(b) Let $f,g \in \mathcal{F}(X,G)$ and let $y \in Y \subset X$. Then $\phi(f*g)(y) = (f*g)(y)$ by the definition of ϕ. This is then $f(y)g(y)$, and then by the definition of ϕ again, we have $f(y) = \phi(f)(y)$ and $g(y) = \phi(g)(y)$, and so have $f(y)g(y) = \phi(f)(y)\phi(g)(y)$. Then by the definition of $*$ on Y, we have $(\phi(f)*\phi(g))(y)$, and so, putting it all together, we have that $\phi(f*g)(y) = (\phi(f)*\phi(g))(y)$, and so ϕ is a homomorphism.

(c) The image of ϕ consists of all functions $f : Y \to G$ which can be obtained by restricting a function $\tilde{f} : X \to G$. Fix $f : Y \to G$ arbitrary. Then define $\tilde{f}$ by $\tilde{f}(x) = \begin{cases} f(x) & x \in Y \\ e_G & x \notin Y \end{cases}$. For all $y \in Y$, we have $f(y) = \tilde{f}(y)$, and so $\phi(\tilde{f}) = f$. Thus, f is in the image of ϕ. As f was arbitrary, this shows that ϕ is surjective, and so $\mathrm{Im}(\phi) = \mathcal{F}(Y,G)$.

The kernel of ϕ consists of those functions $f : X \to G$ which have $\phi(f)$ equal to the function $e_Y : Y \to G$ which maps all $y \in Y$ to e_G. This will be precisely the functions $f : X \to G$ which have $f(y) = e_G$ for all $y \in Y$, by the definition of ϕ.

6 (a) Assume that $H < G$. Then we want to show that $\mathcal{F}(X,H) < \mathcal{F}(X,G)$. Now, for any group, a subset is a subgroup if and only if for all x,y in the subset, xy^{-1} is in it as well. Let $f,g \in \mathcal{F}(X,H)$. Look at $f * g_{inv}$. Fix $x \in X$. Then $(f * g_{inv})(x) = f(x)g_{inv}(x) = f(x)g(x)^{-1}$, and as $f(x)$ and $g(x)$ are in H, which is a subgroup, $f(x)g(x)^{-1} \in H$, and so $f * g_{inv}$ defines a map $X \to H$, thus telling us that $\mathcal{F}(X,H)$ is a subgroup of $\mathcal{F}(X,G)$.

Assume that $\mathcal{F}(X,H)$ is a subgroup of $\mathcal{F}(X,G)$. Let $a,b \in H$. We want to show that $ab^{-1} \in H$. Define $f_a, f_b : X \to H$ by $f_a(x) = a$ and $f_b(x) = b$ for all $x \in X$. Then $f_a, f_b \in \mathcal{F}(X,H)$. Thus, in particular, $f_a * f_{b,inv} \in \mathcal{F}(X,H)$. Now, evaluate on x, and we obtain $(f_a * f_{b,inv})(x) = f_a(x)f_{b,inv}(x) = f_a(x)f_b(x)^{-1} = ab^{-1}$. Now, as $f_a * f_{b,inv} \in \mathcal{F}(X,H)$, anything we get by evaluation is in H, and so $ab^{-1} \in H$. So H is a subgroup of G.

(b) Assume that H is normal in G. We want to show that for all $g \in \mathcal{F}(X,G)$ and all $h \in \mathcal{F}(X,H)$, we have $g * h * g_{inv} \in H$. To do so, let $g \in \mathcal{F}(X,G)$, $h \in \mathcal{F}(X,H)$ and $x \in X$ arbitrary. Then look at $(g * f * g_{inv})$. We must show that this maps into H. Evaluating on x, we obtain $(g * f * g_{inv})(x) = g(x)f(x)g(x)^{-1}$. Now, as $f(x) \in H$ and $g(x) \in G$, with H normal in G, we have $g(x)f(x)g(x)^{-1} \in H$, as desired.

Now, assume that $\mathcal{F}(X,H)$ is normal in $\mathcal{F}(X,G)$. Keep our definition of f_a from above for all $a \in G$. Let $g \in G$ and $h \in H$. We claim that

$ghg^{-1} \in H$. Fix $x \in X$, and look at $f_g * f_h * f_{h,inv}$. As $\mathcal{F}(X,H)$ is a normal subgroup, we have that $f_g * f_h * f_{g,inv}$ is a map $X \to H$. Evaluating on x, we get the element of H $(f_g * f_h * f_{g,inv})(x) = f_g(x)f_h(x)f_g(x)^{-1} = ghg^{-1} \in H$, so H is normal in G.

(c) Assume that H is a normal subgroup of G (and thus, by the previous part, $\mathcal{F}(X,H)$ is a normal subgroup of $\mathcal{F}(X,G)$). Then we claim that there exists $\phi : \mathcal{F}(X,G/H) \to \mathcal{F}(X,G)/\mathcal{F}(X,H)$ which is an isomorphism of groups.

We define ϕ as follows: let $f : X \to G/H$ be any function. Then for each x, $f(x) = gH$ for some $g \in G$. We set $\phi(f)$ to be the class of functions $f' : X \to G$ which contains the function $f'(x) = g$, with g as above. We must prove that ϕ is well-defined, a homomorphism, injective and surjective.

To see that it is well defined, let $f : X \to G/H$, let $x \in X$ and let $g, g' \in G$ such that $f(x) = gH = g'H$. Then there are two candidates for $\phi(f)$, functions with $f'(x) = g$ and $f''(x) = g'$ (defined point by point on X). We must show that they are in the same class modulo $\mathcal{F}(X,H)$. To do so, we must show that $f' * f''_{inv}$ is in the same class as the identity map, because then f' and f'' have the same inverse class. Now, $(f' * f''_{inv})(x) = f'(x)f''(x)^{-1} = gg'^{-1}$. Now, gg'^{-1} is an element of H, as $gH = g'H = Hg' = Hg$ (with the last because H is normal), and we can right multiply by g'^{-1} to obtain the equation $H = Hgg'^{-1} = gg'^{-1}H$. Thus, there exists a function $\alpha : X \to H$ such that $f' * f''_{inv} = \alpha$, and so f' and f'' define the same class. So ϕ is a well defined function.

Next we must check that it is a homomorphism. Let $f, g : X \to G/H$. Then $f * g$ is a function $X \to G/H$. So $\phi(f * g)$ defines a class in the quotient $\mathcal{F}(X,G)/\mathcal{F}(X,H)$. Let $x \in X$. Set $f(x) = aH$ and $g(x) = bH$. Then $(f * g)(x) = abH$. Now, apply ϕ to $f * g$ and we get $\phi(f * g)(x)$ is a function $\alpha : X \to G$ such that $\alpha(x) = ab$. We want to show that α is in the same class as $\phi(f) * \phi(g)$. Now, $(\phi(f) * \phi(g))(x) = \phi(f)(x)\phi(g)(x) = ab$, and so for all $x \in X$, we have $\phi(f)(x)\phi(g)(x) = \alpha(x) = \phi(f * g)(x)$, and so ϕ is a homomorphism.

Now we show that it is injective. Let $f : X \to G/H$ such that $\phi(f)$ is a function $X \to H$. Let $x \in X$, then $f(x) = gH$ for some $g \in G$, and $\phi(f)(x) = g$. Now, $\phi(f)$ is a function $X \to H$ if and only if $g \in H$, and so $f(x) = H$ in the first place, which is the same thing as f being the identity element in $\mathcal{F}(X,G/H)$, so the kernel is trivial, and ϕ is injective.

Finally, we want to show that ϕ is surjective. Let $f : X \to G$ by any function. Define $\tilde{f} : X \to G/H$ by for all $x \in X$, $\tilde{f}(x) = f(x)H$. Then $\phi(\tilde{f})(x) = f(x)$ for all $x \in X$, and so $\phi(\tilde{f}) = f$. Thus, ϕ is surjective, and so is an isomorphism.

8 First, we will show that if $|X| > 1$ then $\mathcal{F}(X,R)$ cannot be a skew field,

and then we will show that, if $|X| = 1$, then $\mathcal{F}(X, R)$ is a skew field if and only if R is, which solves the problem.

Assume that $|X| > 1$. Let $x, y \in X$ with $x \neq y$. Then define $f \in \mathcal{F}(X, R)$ by $f(x) = 1_R$ and $f(a) = 0$ for all $a \in X$ with $a \neq x$ and define $g(y) = 1_R$ and $g(a) = 0$ for all $a \in X$ not equal to a. Then for all $a \in X$, we have $(f * g)(a) = f(a)g(a) = 0$, because at least one of them is zero. Thus, $\mathcal{F}(X, R)$ has zero divisors and cannot be a skew field.

Now assume that $|X| = 1$. Denote the single element of X by x. We will in fact prove a stronger statement: $\mathcal{F}(X, R)$ is isomorphic to R, regardless of the properties of R. Define a map $\gamma : \mathcal{F}(X, R) \to R$ by for each $f : X \to R$, $\gamma(f) = f(x)$. We first prove that this is a homomorphism of rings with identity.

Let $f, g \in \mathcal{F}(X, R)$. Then $\gamma(f \oplus g) = (f \oplus g)(x) = f(x) + g(x) = \gamma(f) + \gamma(g)$ and $\gamma(f \odot g) = (f \odot g)(x) = f(x)g(x) = \gamma(f)\gamma(g)$, so it is a ring homomorphism. Additionally, if e is the multiplicative identity, then it must have $e(x) = 1_R$, and so $\gamma(e) = 1_R$, so it preserves the identity. Now, we must show surjective and injective.

To see that γ is surjective, let $r \in R$. Then we can define a function $f : X \to R$ by $f(x) = r$ for the only $x \in X$. Then, $\gamma(f) = f(x) = r$, and so γ is surjective. To see that it is injective, let f be any function such that $\gamma(f) = 0$. Then $f(x) = 0$, but the only element of X is x, so f is the constant function mapping all of X to zero, which is the additive identity of $\mathcal{F}(X, R)$. Thus, the kernel is trivial and γ is injective, and so is an isomorphism.

As $\mathcal{F}(X, R)$ is isomorphic to R, certainly R is a skew field if and only if $\mathcal{F}(X, R)$ is.