Math 501 - Differential Geometry
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8. THE FARY-MILNOR THEOREM

The curvature κ of a smooth curve in 3-space is ≥ 0 by definition, and its integral w.r.t. arc length, $\int \kappa(s)\, ds$, is called the ***total curvature*** of the curve.

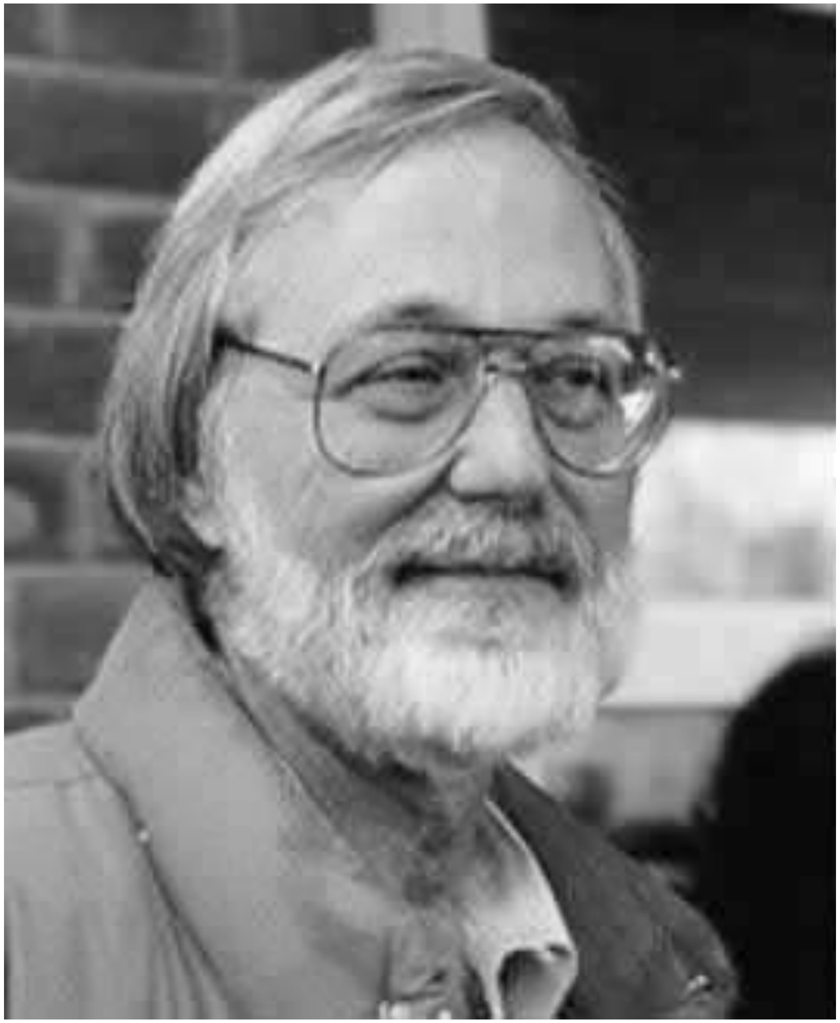
According to Fenchel's Theorem, the total curvature of any simple closed curve in 3-space is $\geq 2\pi$, with equality if and only if it is a plane convex curve.

According to the Fary-Milnor Theorem, if the simple closed curve is knotted, then its total curvature is $> 4\pi$.

In 1949, when Fary and Milnor proved this celebrated theorem independently, Fary was 27 years old and Milnor, an undergraduate at Princeton, was 18.

In these notes, we'll prove Fenchel's Theorem first, and then the Fary-Milnor Theorem.

Istvan Fary (1922-1984) in 1968



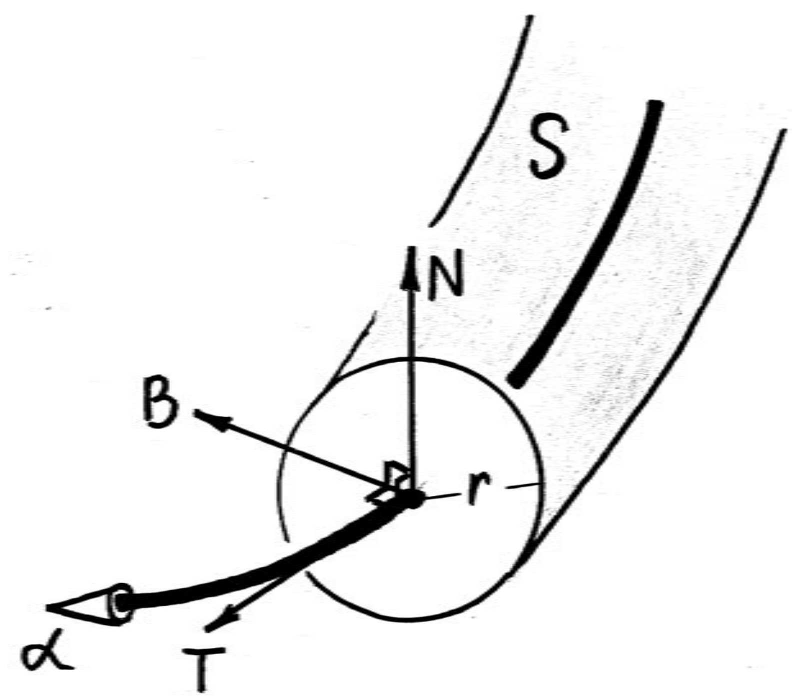
John Milnor (1931 -)

Fenchel's Theorem.

We consider a smooth closed curve α: [0, L] $\rightarrow$ R^3 , parametrized by arc-length. In order to make use of the associated Frenet frame, we assume that the curvature κ is never zero.

FENCHEL's THEOREM (1929?). *The total curvature of a smooth simple closed curve in 3-space is* $\geq 2\pi$ *, with equality if and only if it is a plane convex curve.*

Proof. In this first step, we will show that the total curvature is $\geq 2\pi$. We start with the smooth simple closed curve α , construct a tubular neighborhood of radius r about it, and focus on the toroidal surface S of this tube.



Taking advantage of our assumption that the curvature κ of our curve α never vanishes, we have a well-defined Frenet frame $T(s)$, $N(s)$, $B(s)$ at each point $\alpha(s)$.

We recall the Frenet equations:

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N . \end{aligned}$$

The surface S bounding our tubular neighborhood of radius r is parametrized by

$$X(s, v) = \alpha(s) + r \cos v \, N(s) + r \sin v \, B(s) .$$

Then

$$X_s = \alpha' + r \cos v \, N' + r \sin v \, B'$$

$$= T + r \cos v \, (-\kappa \, T + \tau \, B) + r \sin v \, (-\tau \, N)$$

$$= (1 - r \, \kappa \cos v) \, T - r \, \tau \sin v \, N + r \, \tau \cos v \, B \, ,$$

where κ , T , N and B all depend on s , and

$$X_v = - r \sin v \, N + r \cos v \, B \, .$$

Thus

$$X_s \times X_v = -r\cos v(1 - r\kappa\cos v)N - r\sin v(1 - r\kappa\cos v)B$$

$$|X_s \times X_v| = r(1 - r\kappa\cos v).$$

We choose the radius r of our tube sufficiently small so that the tube is smoothly embedded in R^3, and in particular so that $r < 1/\max\kappa$, which guarantees that

$$|X_s \times X_v| = r(1 - r\kappa\cos v) > 0.$$

Next we calculate the coefficients of the first fundamental form of the surface S .

$$E = \langle X_s , X_s \rangle = (1 - r\,\kappa \cos v)^2 + r^2 \tau^2$$

$$F = \langle X_s , X_v \rangle = r^2 \tau$$

$$G = \langle X_v , X_v \rangle = r^2$$

$$EG - F^2 = r^2 (1 - r\,\kappa \cos v)^2$$

$$(EG - F^2)^{1/2} = r\,(1 - r\,\kappa \cos v) = |X_s \times X_v| \,.$$

The inward pointing unit normal vector to the surface S is

$$M = (X_s \times X_v) / |X_s \times X_v| = -\cos v\, N - \sin v\, B .$$

Note that we are using the letter "M" for the surface normal, since we have already used "N" for the principal normal to our curve α . Then

$$\begin{aligned} M_s &= -\cos v\, N' - \sin v\, B' \\ &= -\cos v\,(-\kappa\, T + \tau\, B) - \sin v\,(-\tau\, N) \\ &= \kappa \cos v\, T + \tau \sin v\, N - \tau \cos v\, B , \\ M_v &= \qquad\qquad \sin v\, N - \cos v\, B . \end{aligned}$$

From this we get

$$M_s \times M_v = \kappa \cos^2 v\, N + \kappa \sin v \cos v\, B$$

$$|M_s \times M_v| = \kappa \cos v\,.$$

We compare

$$X_s \times X_v = -\, r\cos v(1 - r\kappa\cos v)N - r\sin v(1 - r\kappa\cos v)B$$

$$= -r\,(1 - r\,\kappa\,\cos v)\,(\cos v\, N + \sin v\, B)\,,$$

$$M_s \times M_v = \kappa \cos^2 v\, N + \kappa \sin v \cos v\, B$$

$$= \kappa \cos v\,(\cos v\, N + \sin v\, B)\,.$$

We see that

$$M_s \times M_v = \{-\kappa \cos v / [r (1 - r \kappa \cos v)]\} X_s \times X_v .$$

Since Gaussian curvature K is the explosion factor for oriented area under the Gauss map $M: S \rightarrow S^2$, we have

$$K = -\kappa \cos v / [r (1 - r \kappa \cos v)] .$$

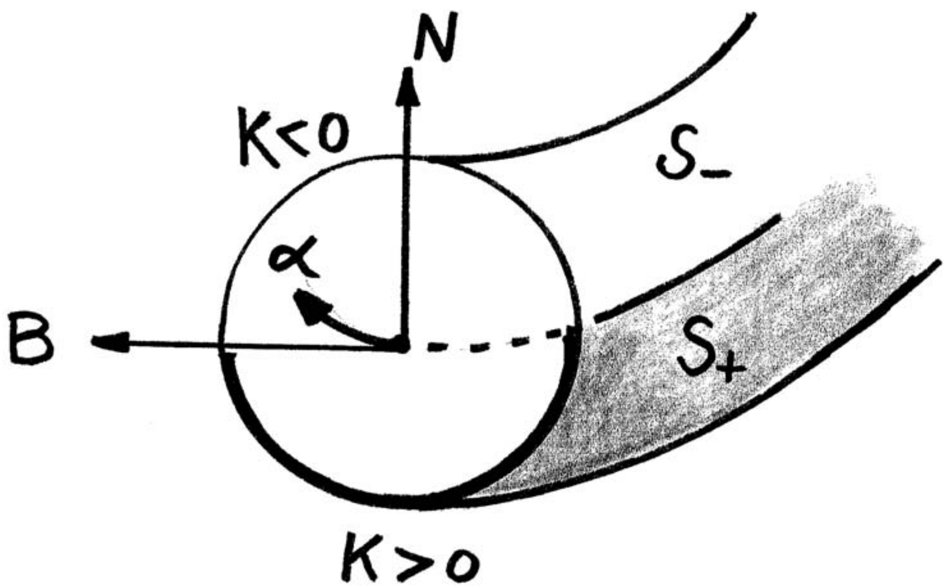
Problem. Double check this value of K by computing the coefficients of the second fundamental form:

$$e = -\kappa \cos v\,(1 - r\kappa \cos v) + r\tau^2$$
$$f = r\tau$$
$$g = r\,.$$

Then use the formula $K = (eg - f^2) / (EG - F^2)$ to compute the Gaussian curvature.

Focus on the formula for the Gaussian curvature of the surface S bounding the tubular neighborhood of radius r about our curve α :

$$K = -\kappa \cos v / [r(1 - r\kappa \cos v)] .$$



We note that the Gaussian curvature K is ≥ 0 when $\cos v \leq 0$, that is when $\pi/2 \leq v \leq 3\pi/2$, which appears as the bottom S_+ of S in the figure.

$K \leq 0$ on the remaining (closed) half S_- of S.

By the Gauss-Bonnet Theorem, the total Gaussian curvature of the surface S is zero,

$$\int_S K \, d(\text{area}) = 0 ,$$

because S is a torus with Euler characteristic $\chi(S) = 0$.

By contrast, let's compute the integral of the Gaussian curvature only over the region S_+ where $K \geq 0$.

$$\int_{S+} K \ d(\text{area}) = \int_{S+} K \ (EG - F^2)^{1/2} \ ds \, dv$$

$$= \int_{S+} (-\kappa\cos v \, / \, [r(1 - r\kappa\cos v)]) \ r \, (1 - r\kappa\cos v) \ ds \, dv$$

$$= \int_{S+} - \kappa \cos v \ ds \, dv$$

$$= \int_{v=\pi/2}^{3\pi/2} (- \cos v) \, dv \ \int_{s=0}^{L} \kappa \ ds$$

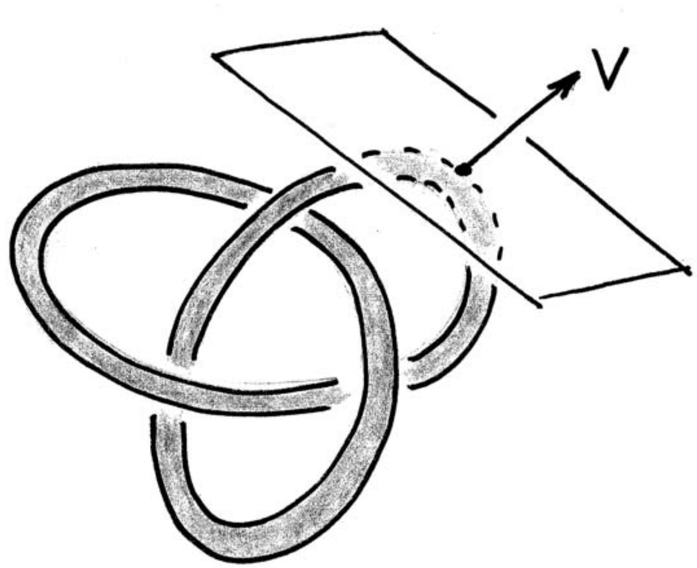
$$= 2 \int_{s=0}^{L} \kappa \ ds \, .$$

Thus the total Gaussian curvature of the "positive half" S_+ of the toroidal surface S equals twice the total curvature of the curve α .

It is easy to see that the Gauss image $M(S_+)$ covers S^2 at least once, as follows.

Given any unit vector $V \in S^2$, take a plane in R^3 which is orthogonal to V, far away from α and containing α in its negative half-space.

Then move this plane towards α, always keeping it orthogonal to V. When the moving plane first touches the boundary S of our tubular neighborhood of α, it will do so at a point of S where $K \geq 0$.



This shows that $M(S_+) = S^2$, and hence that

$$\int_{S+} K \, d(\text{area}) \geq 4\pi$$

Since $$\int_{S+} K \, d(\text{area}) = 2 \int_{s=0}^{L} \kappa \, ds ,$$

we conclude that $$\int_{s=0}^{L} \kappa \, ds \geq 2\pi .$$

Plane convex curves. Suppose next that our simple closed curve α is a convex curve in the plane.

We have seen earlier in this course that its total curvature is 2π , but let us see this again by the present methods.

To that end, consider the boundary S of our tubular nbhd of α. The points on S with a given value of s form a circle of radius r , whose image under the Gauss map $M: S \rightarrow S^2$ is a great circle Γ_s .

Denote by Γ_s^+ the closed semi-circle of Γ_s corresponding to points where $K \geq 0$.

Since α is a plane curve, all the semi-circles Γ_s^+ run between the north and south poles on S^2.

Since α is convex, and in fact, strictly convex because we are assuming its curvature never vanishes, the various semi-circles Γ_s^+ meet only at the north and south poles.

It follows that the Gaussian image $M(S_+)$ covers S^2 just once (except for the overlaps at the two poles) and hence that

$$\int_{S+} K \ d(\text{area}) = 4\pi .$$

It follows that the total curvature of α satisfies

$$\int_{s=0}^{L} \kappa \ ds = 2\pi .$$

Curves with total curvature 2π . Assume now that the total curvature $\int_{s=0}^{L} \kappa \; ds$ of α is 2π .

We must show that α is a convex plane curve.

We know that

$$\int_{S+} K \; d(\text{area}) = 4\pi \quad \text{and} \quad \int_{S-} K \; d(\text{area}) = -4\pi .$$

Thus, the area of the Gauss image $M(S)$ of S is 8π , counting multiplicity but not orientation.

Problem. Let $T(\alpha) \subset S^2$ denote the curve traced on S^2 by the unit tangent vectors $T(s) = \alpha'(s)$ to the curve α .

(a) Show that the length of $T(\alpha)$ is the same as the total curvature of α .

(b) Show that area of the Gauss image $M(S)$, counted as above, is four times the length of $T(\alpha)$.

(c) Show that the curve $T(\alpha)$ meets every great circle on S^2 .

Problem. Show that if the total curvature of α is 2π , then all the semi-circles Γ_s^+ have the same endpoints.

Now, since all the semi-circles $\Gamma_s{}^+$ on S^2 have the same endpoints, we see from an earlier the figure that all the points $\alpha(s)$ on the curve α have the same binormal vector $B = B(s)$.

Problem. Show that if all the points on a curve in 3-space have the same binormal vector B , then the curve lies in a plane orthogonal to B .

Now we know that our curve α of total curvature 2π lies in a plane.

If we let $\kappa(s)$ denote the signed curvature of α as a plane curve, then we have

$$2\pi = \int_0^L |\kappa(s)|\, ds \geq \int_0^L \kappa(s)\, ds = 2\pi .$$

It follows that the signed curvature $\kappa(s) > 0$, and therefore that α is a convex plane curve.

This completes the proof of Fenchel's Theorem.

Problem. Fenchel's Theorem was proved under the hypothesis that the curve α had nowhere vanishing curvature. Show how to get rid of this hypothesis.

The Fary-Milnor Theorem.

FARY-MILNOR THEOREM. *The total curvature of a smooth simple closed curve in 3-space which is knotted is* $> 4\pi$.

Proof.

We'll use the same notation as in the proof of Fenchel's Thm, $\alpha: [0, L] \to R^3$ is a smooth simple closed curve parametrized by arc length, and with nowhere vanishing curvature, S is the boundary of a tubular neighborhood of α of radius r , and S_+ is the portion of S where its Gaussian curvature $K \geq 0$.

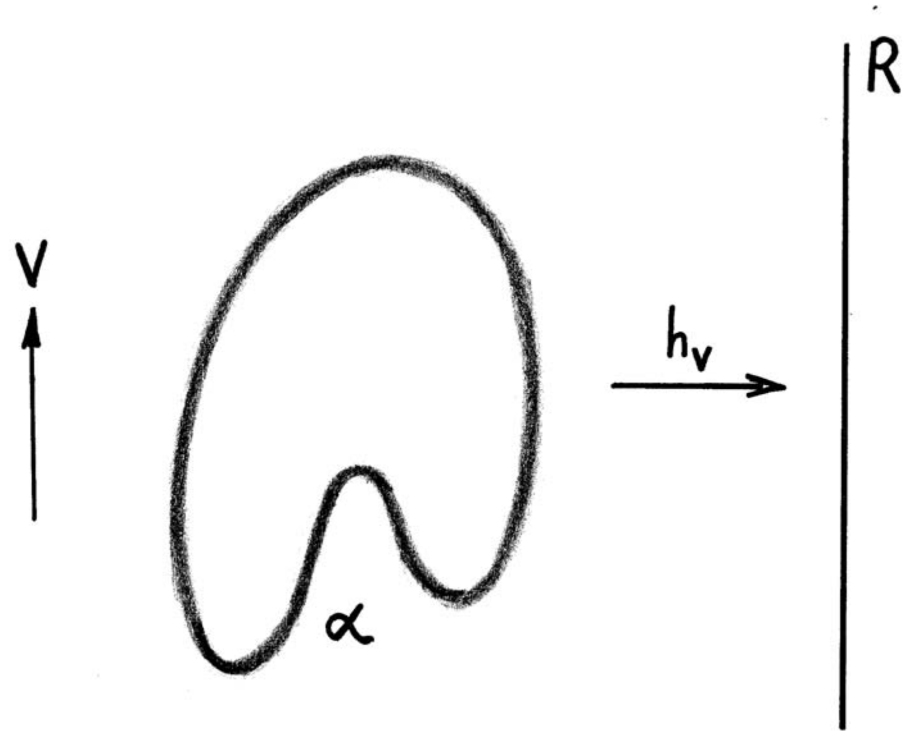
At each point $\alpha(s)$ we have, thanks to the hypothesis of nowhere vanishing curvature, the orthonormal Frenet frame

$$T(s)\ ,\ \ N(s)\ ,\ \ B(s)\ .$$

Let V be a unit vector which is different from all the binormal vectors $B(s)$, $s \in [0, L]$, and their negatives.

Let $h_V: [0, L] \to R$ be the ***height function*** of α in the direction of V,

$$h_V(s) = \langle \alpha(s)\ ,\ V \rangle\ .$$



Since $h_V'(s) = \langle \alpha'(s), V \rangle = \langle T(s), V \rangle$,
we see that s is a critical point of h_V precisely when the tangent vector $T(s)$ is orthogonal to V.

We claim that the precaution of choosing V different from all the binormal vectors $B(s)$ insures that all critical points of h_V are ***nondegenerate*** , that is, $h_V''(s) \neq 0$.

To see this, note that

$$h_V''(s) = < \alpha''(s) , V > = < \kappa(s) N(s) , V > .$$

Now if s is a critical point of h_V , then $\alpha'(s) = T(s)$ must be orthogonal to V , which means that V is some linear combination of $N(s)$ and $B(s)$.

Since $V \neq \pm B(s)$, we must have $< N(s), V > \neq 0$.

Since $\kappa(s) \neq 0$, we must have $< \kappa(s) N(s), V > \neq 0$.

Thus $h_V''(s) \neq 0$.

Hence all critical points of h_V are nondegenerate maxima or nondegenerate minima.

In particular, there are only finitely many of them.

Conclusion of the proof.

Suppose that the total curvature of α is $< 4\pi$. Then

$$\int_{S+} K \ d(\text{area}) = 2 \int_0^L \kappa(s) \ ds < 8\pi .$$

CLAIM. *In such a case, there is some unit vector* V *with* $V \neq \pm B(s)$ *for all* $s \in [0, L]$, *such that the function* h_V *has exactly two critical points.*

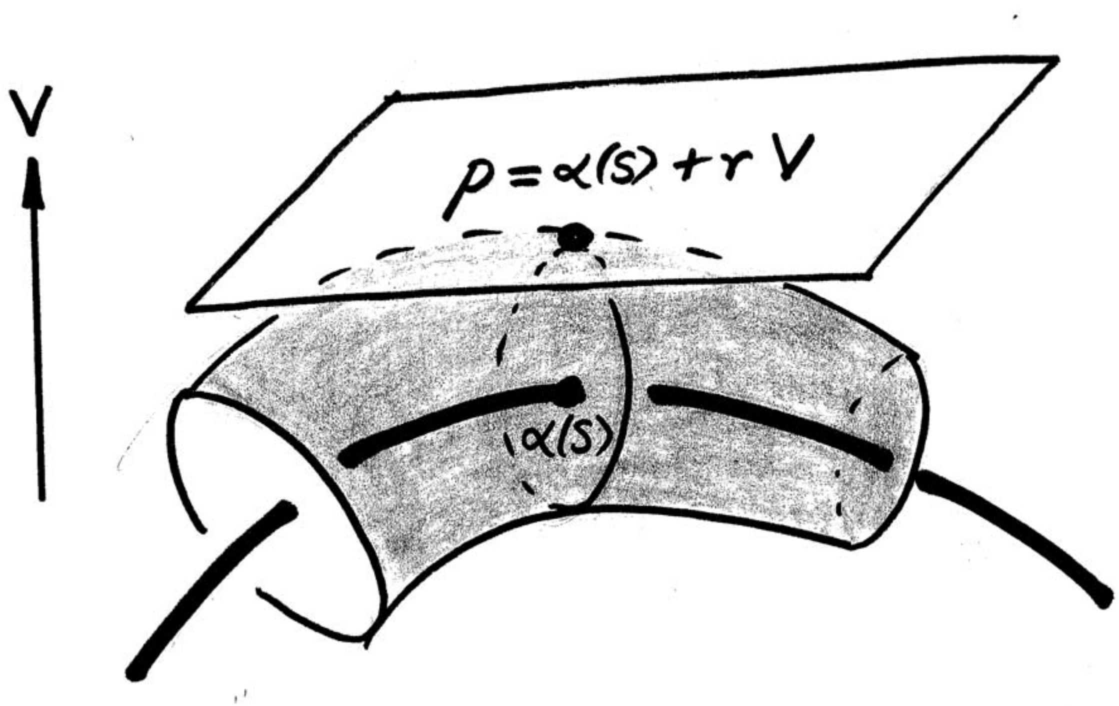
Proof of Claim. Suppose to the contrary that each such h_V has at least three critical points. Since all the critical points are nondegenerate maxima and minima, these must alternate around α and hence there must be at least four critical points.

Suppose h_V has a nondegenerate local maximum at s .

Then the tangent plane to the toroidal surface S at the point $p = \alpha(s) + rV$ is orthogonal to V , and a neighborhood of p on S lies to one side of this plane.

Hence the Gaussian curvature of S at p , $K(p) \geq 0$.

Since $V \neq \pm B(s)$, we actually have $K(p) > 0$.



Note that the image M(p) of p under the Gauss map $M\colon S \rightarrow S^2$ satisfies $M(p) = V$.

Thus the two local maxima of h_v contribute together two points p and q satisfying

$$M(p) = V = M(q) , \quad K(p) > 0 \quad \text{and} \quad K(q) > 0 .$$

Since the set of unit vectors V which avoid the points $\pm B(s)$ for $s \in [0, L]$ is dense in S^2 , we see that the Gauss map $M: S_+ \to S^2$ covers S^2 at least twice, and hence

$$\int_{S+} K \, d(\text{area}) \geq 8\pi .$$

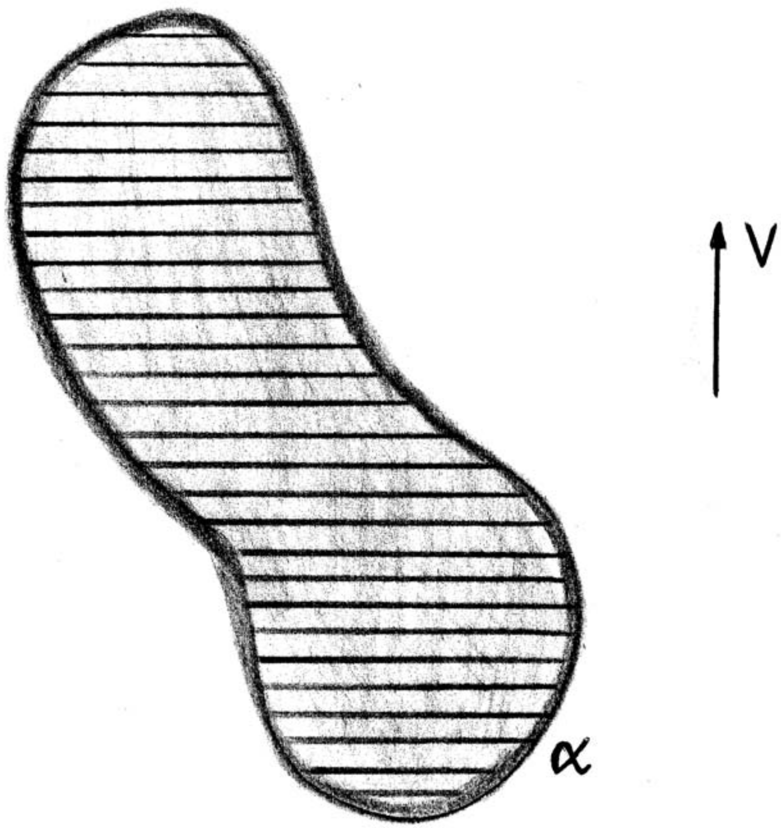
Since we were assuming that $\int_{S+} K \, d(\text{area}) < 8\pi$, this contradiction proves the claim.

Unknotting a curve with total curvature $< 4\pi$.

Suppose α is a simple closed curve in 3-space with total curvature $< 4\pi$. Then by the Claim proved above, there is a unit vector $V \neq \pm B(s)$ for all $s \in [0, L]$, such that the height function $h_V : [0, L] \to R$ has just one maximum and one minimum.

Begin with a plane orthogonal to V which touches the curve α at its unique highest point. As we lower this plane, keeping it orthogonal to V , it will begin to cut α in two points.

Connect these two points by a line segment in the given plane.



As we continue to lower the plane, we continue to intersect α in two points, varying continuously, until we come to the minimum point on α .

The union of the line segments drawn in this way is a topological disk bounded by α , which shows that α is unknotted.

Thus a knotted simple closed curve in 3-space has total curvature $\geq 4\pi$.

Problem. How do you refine this argument to prove that a knotted simple closed curve in 3-space has total curvature $> 4\pi$?

Problem. How do you avoid the hypothesis that the curvature of α is nowhere vanishing ?