Math 501 - Differential Geometry
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# 5. THE ISOPERIMETRIC PROBLEM

**Theorem.** ***Let C be a simple closed curve in the plane with length L and bounding a region of area A . Then***

$$L^2 \geq 4\pi A \, ,$$

***with equality if and only if C is a circle.***

Thus, among all simple closed curves in the plane with a given length, the circle bounds the largest area.

The proof we'll discuss here comes from the book "What is Mathematics?" by Richard Courant and Herbert Robbins, and is credited by them to Jakob Steiner (1796-1863), who was Riemann's teacher. Another proof can be found in our text by do Carmo on pages 31 - 35.

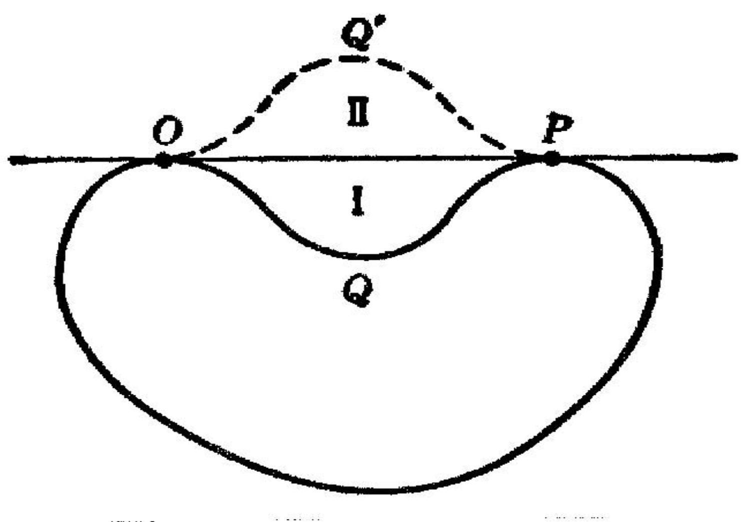
We'll take for granted the ***Jordan Curve Theorem***, which says that a simple closed curve in the plane divides the plane into two regions, one compact and one noncompact, and is the common boundary of both regions. When we talk of the region bounded by a simple closed curve in the plane, we'll always mean the compact region.

**Proof.** We'll begin the proof of the Isoperimetric Theorem with the assumption that a solution exists, that is, that there is a simple closed curve C of given length L bounding a region of maximum area.

**(1)** We claim that the curve C must be ***convex***, in the sense that any line segment joining two points of C must lie entirely in the closure of the region bounded by C .

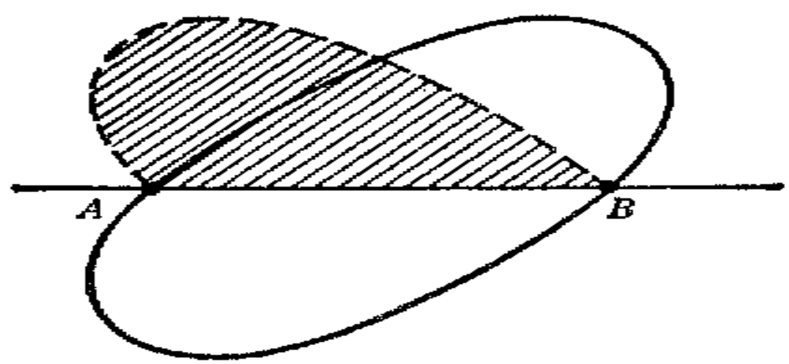
We see this as follows. If C were not convex, then we could draw a segment OP between two points of C which lies entirely (except for its endpoints) outside of C .

Reflecting the appropriate arc of C between O and P in this line would produce another curve of the same length but bounding a larger area, as in the figure below.




Hence C must already be convex.

**(2)** Now choose two points, A and B , dividing our solution curve C into arcs of equal length. Then the line segment AB must also divide the region bounded by C into two parts of equal area. Otherwise, the part of greater area could be reflected in AB to give another curve of the same length L bounding a region of area greater than that bounded by C .

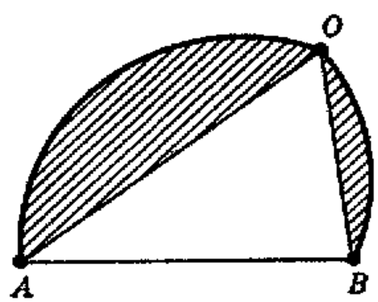


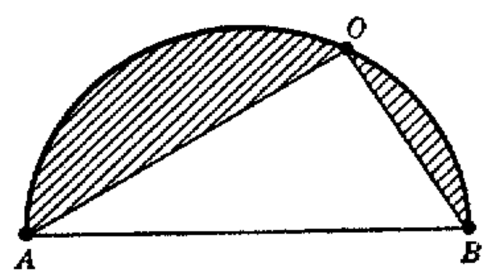
It follows that half of our solution curve $C$ must solve the following problem:

To find the arc of length $L/2$ with endpoints $A$ and $B$ lying anywhere on a straight line, such that the arc together with the segment $AB$ encloses a region of maximum area.

**(3)** Now we'll show that the solution to this new problem is a semi-circle, so that the solution to the original problem must be a full circle.

Suppose the arc AOB shown at the left below solves this new problem. It is sufficient to show that every inscribed angle, such as the one at O, is a right angle, for this will prove that our arc is a semi-circle.



Suppose, to the contrary, that the angle at O is not $90^o$ .
Then we can view the arc as hinged at O , and either open it or close it so as to make the angle at O exactly $90^o$ .

This will not change the length of the arc, nor change the two shaded areas, but will increase the triangular area, and therefore increase the total area enclosed by the arc AOB and the line segment AB .

But this was already maximized by the original arc, so its inscribed angle at O is $90^o$ and the arc itself a semi-circle.

**(4) Where are we?** So far, we have shown that ***if*** there is a simple closed curve C of given length L bounding a region of maximum area, then that curve must be a circle.

Now we will prove that a maximizer does exist, and present this as the full isoperimetric theorem:

> ***If* C *is a smooth simple closed curve in the plane with length* L *and bounding a region of area* A *,* *then* $\mathbf{L^2 \geq 4\pi A}$ *, with equality iff* C *is a circle.***

We'll do this by approximating C by a polygonal curve, and then applying the ideas of Steiner discussed above to polygonl curves.

**(5) LEMMA. Among all 2n-sided polygons with the same length L , the regular 2n-gon has the largest area.**

***Remark.*** Existence of a maximizer among such 2n-gons is evident, since the vertices may be restricted to a compact region of the plane, for example, a disk of radius L . Hence the set of such 2n-gons is itself compact, and since the area is a continuous function, it is certainly maximized.

Exactly as in Steiner's proof, it follows that the maximizer must be convex.

Suppose that two adjacent edges AB and BC had different lengths. Then we could cut off triangle ABC from our polygon, and replace it with an isosceles triangle AB'C in which AB' + B'C = AB + BC , and which has a larger area (prove this!).

It follows that on the area-maximizing 2n-gon of length L , all the edges are equal.

Following Steiner's pattern, we cut such a maximizer into two polygonal n-gons, each of length L/2 , and then show that each is inscribed in a semi-circle through its endpoints.

It follows that the entire maximizer is inscribed in a circle, and hence (since its edges are all equal), is regular.

This completes the proof of the Lemma.

**Problem 1.** Show that if $P$ is a regular 2n-gon of length $L$ and area $A$, then $L^2 \geq 4\pi A$. Conclude that for *any* polygon of length $L$ and area $A$, we have $L^2 \geq 4\pi A$.

**(6) Completion of the proof of the isoperimetric theorem.**

Let $C$ be a smooth, regular simple closed curve in the plane, with length $L$ and bounding a region of area $A$ .

For each $\varepsilon > 0$ , we can inscribe in $C$ a polygon $P_\varepsilon$ whose length $L_\varepsilon$ satisfies $|L_\varepsilon - L| < \varepsilon$ and whose area $A_\varepsilon$ satisfies $|A_\varepsilon - A| < \varepsilon$ .

For the polygon $P_\varepsilon$ we already have the isoperimetric inequality $L_\varepsilon^2 \geq 4\pi A_\varepsilon$ .

Now let $\varepsilon \rightarrow 0$ and we get the isoperimetric inequality for the given curve $C$, $L^2 \geq 4\pi A$.

Since the circle of the same length $L$ bounds a region of area $A_0$ satisfying $L^2 = 4\pi A_0$, we know that $A \leq A_0$.

Thus we know that a circle maximizes enclosed area among all smooth regular simple closed curves of the same length. So the maximizer exists.

Then by parts (1) - (3) of this proof, there are no other maximizers, completing the proof of the Isoperimetric Theorem.