5.6

T/F.2. True. If A is upper or lower diagonal, to make $\det(A - \lambda I) = 0$, we need product of the main diagonal elements of $A - \lambda I$ to be 0, which means λ is one of the main diagonal elements of A.

T/F.4. False. For example $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

T/F.8. True. The characteristic polynomial has degree n, which has n complex solutions taking into account of multiplicity.

Prob.8. The transformation T projects any vector onto its y component. From its geometrical meaning, $T\mathbf{i} = T\mathbf{k} = 0$ and $T\mathbf{j} = \mathbf{j}$. Therefore we have two eigenvalues 0 and 1. Eigenvalue 0 has two eigenvectors $r\mathbf{i}$ and $s\mathbf{k}$. Eigenvalue 1 has eigenvector $t\mathbf{j}$. Here r, s, t are arbitrary nonzero real numbers.

Prob.16. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $A - \lambda I = \begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$,

$$\det(A - \lambda I) = (3 - \lambda)[(2 - \lambda)^2 - 1].$$

The characteristic has two real solution $\lambda = 3$ and $\lambda = 1$. $\lambda = 3$'s eigenvector satisfies

$$(A - 3I)v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} v = 0.$$

There is one linearly independent solution to this equation, which can be taken to be $v_3 = r(0, 1, -1)$. For $\lambda = 1$ we have

$$(A - I)v = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} v = 0.$$

We can solve its eigenvector to be $v_1 = s(0, 1, 1)$. Here r, s are arbitrary nonzero numbers.

Prob.32. First we want to solve $\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$ for c_1, c_2, c_3. This is equivalent to

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}.$$

We do a series of row operation on augmented matrix as follows.

$$\begin{bmatrix} 1 & 2 & -1 & 5 \\ -1 & 1 & -1 & 0 \\ 1 & 3 & 2 & 3 \end{bmatrix} \xrightarrow{A_{12}(1), A_{13}(-1)} \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 3 & -2 & 5 \\ 0 & 1 & 3 & -2 \end{bmatrix} \xrightarrow{P_{23}} \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & -2 & 5 \end{bmatrix}$$
$$\xrightarrow{A_{23}(-3)} \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -11 & 11 \end{bmatrix} \xrightarrow{M_{23}(-1/11)} \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Therefore $c_3 = -1$, $c_2 = -2 - 3c_3 = 1$ and $c_1 = 5 - 2c_2 + c_3 = 2$. Therefore

$$\mathbf{v} = 2\mathbf{v_1} + \mathbf{v_2} - \mathbf{v_3}.$$

It follows that

$$A\mathbf{v} = 2A\mathbf{v_1} + A\mathbf{v_2} - A\mathbf{v_3} = 4\mathbf{v_1} - 2\mathbf{v_2} - 3\mathbf{v_3} = (3, -3, -8).$$

Prob.38. Since $\det(M) = \det(M^T)$ for any square matrix M, we have

$$\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I).$$

This means A and A^T have the same characteristic equation, hence they have the same eigenvalues.

5.8

T/F.2. True. If $S^{-1}AS = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ and A is invertible, then none of the λ_i's is zero. We have

$$S^{-1}A^{-1}S = \operatorname{diag}(\lambda_1^{-1}, \cdots, \lambda_n^{-1}).$$

T/F.6. True. If $S^{-1}AS = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ then

$$S^{-1}A^2S = \operatorname{diag}(\lambda_1^2, \cdots, \lambda_n^2).$$

Prob.8. Since A has rank 1, it only has one non-zero eigenvalue. We can also see this from direct calculation as follows. $A = \begin{bmatrix} -2 & 1 & 4 \\ -2 & 1 & 4 \\ -2 & 1 & 4 \end{bmatrix}$,

$$A - \lambda I = \begin{bmatrix} -2-\lambda & 1 & 4 \\ -2 & 1-\lambda & 4 \\ -2 & 1 & 4-\lambda \end{bmatrix},$$

$$\det(A - \lambda I) = \det \begin{bmatrix} -2-\lambda & 1 & 4 \\ \lambda & -\lambda & 0 \\ \lambda & 0 & -\lambda \end{bmatrix} = \det \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = (3-\lambda)\lambda^2.$$

The characteristic has two real solutions $\lambda = 3$ and $\lambda = 0$. $\lambda = 3$'s eigenvector satisfies

$$(A - 3I)v = \begin{bmatrix} -5 & 1 & 4 \\ -2 & -2 & 4 \\ -2 & 1 & 1 \end{bmatrix} v = 0.$$

There is one linearly independent solution to this equation, which can be taken to be $v_3 = (1, 1, 1)$. For $\lambda = 0$ we have

$$Av = \begin{bmatrix} -2 & 1 & 4 \\ -2 & 1 & 4 \\ -2 & 1 & 4 \end{bmatrix} v = 0.$$

We can solve its eigenvectors to be $v_1 = (0, 4, -1)$ and $v_2 = (2, 0, 1)$.
Now we take

$$S = [v_1, v_2, v_3] = \begin{bmatrix} 0 & 2 & 1 \\ 4 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

We have

$$AS = S\,\mathrm{diag}(0, 0, 3), \quad \text{or} \quad S^{-1}AS = \mathrm{diag}(0, 0, 3).$$

Remark. In solving the eigenvectors, you might end up getting something totally different from my answer, that's completely normal. As long as your answer and mine are linearly dependent on each other, it's fine.

Prob.10. We do this problem using the standard procedure just as in the last one.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 3 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix}, \; A - \lambda I = \begin{bmatrix} 4-\lambda & 0 & 0 \\ 3 & -1-\lambda & -1 \\ 0 & 2 & 1-\lambda \end{bmatrix},$$

$$\det(A - \lambda I) = (4-\lambda)[(-1-\lambda)(1-\lambda) + 2] = (4-\lambda)(\lambda^2 + 1).$$

The characteristic has one real solution $\lambda = 4$ and two imaginary solutions $\lambda = \pm i$. $\lambda = 4$'s eigenvector satisfies

$$(A - 4I)v = \begin{bmatrix} 0 & 0 & 0 \\ 3 & -5 & -1 \\ 0 & 2 & -3 \end{bmatrix} v = 0.$$

We can solve its eigenvector to be $v_1 = (17, 9, 6)$. For $\lambda = i$ we have

$$(A - iI)v = \begin{bmatrix} 4-i & 0 & 0 \\ 3 & -1-i & -1 \\ 0 & 2 & 1-i \end{bmatrix} v = 0.$$

We can solve its eigenvector to be $v_2 = (0, 1, -1-i)$. And for $\lambda = -i$ we have

$$(A - iI)v = \begin{bmatrix} 4+i & 0 & 0 \\ 3 & -1+i & -1 \\ 0 & 2 & 1+i \end{bmatrix} v = 0.$$

We can solve its eigenvector to be $v_3 = (0, 1, -1+i)$.
Now we take

$$S = [v_1, v_2, v_3] = \begin{bmatrix} 17 & 0 & 0 \\ 9 & 1 & 1 \\ 6 & -1-i & -1+i \end{bmatrix}.$$

We have

$$AS = S\,\mathrm{diag}(4, i, -i), \quad \text{or} \quad S^{-1}AS = \mathrm{diag}(4, i, -i).$$

Prob.22. If $A = SDS^{-1}$,

$$A^2 = SDS^{-1}SDS^{-1} = SDDS^{-1} = SD^2S^{-1}.$$

And for any positive integer k,

$$A^k = SDS^{-1}SDS^{-1} \cdots SDS^{-1} = SDD \cdots DS^{-1} = SD^kS^{-1}.$$

The equality holds since all the S^{-1} and S in between D's cancel each other.

Prob.24. We need to diagonalize $A = \begin{bmatrix} -7 & -4 \\ 18 & 11 \end{bmatrix}$ first. $A - \lambda I = \begin{bmatrix} -7-\lambda & -4 \\ 18 & 11-\lambda \end{bmatrix}$.
$\det(A - \lambda I) = (-7-\lambda)(11-\lambda) + 72 = (\lambda+1)(\lambda-5)$. A has two eigenvalues -1 and 5. When $\lambda = -1$,

$$(A - \lambda I)v = \begin{bmatrix} -6 & -4 \\ 18 & 12 \end{bmatrix} v = 0$$

has eigenvector solution $v_1 = (2, -3)$. When $\lambda = 5$,

$$(A - \lambda I)v = \begin{bmatrix} -12 & -4 \\ 18 & 6 \end{bmatrix} v = 0$$

has eigenvector solution $v_2 = (1, -3)$. Therefore we can take

$$S = [v_1, v_2] = \begin{bmatrix} 2 & 1 \\ -3 & -3 \end{bmatrix},$$

and

$$D = \mathrm{diag}(-1, 5).$$

It follows from the prob 22, 23 that

$$A^3 = S\,\mathrm{diag}((-1)^3, 5^3)\,S^{-1} = \begin{bmatrix} -2 & 5^3 \\ 3 & -3 \times 5^3 \end{bmatrix} \begin{bmatrix} 1 & 1/3 \\ -1 & -2/3 \end{bmatrix} = \begin{bmatrix} -2 - 5^3 & -\frac{2(1+5^3)}{3} \\ 3(1+5^3) & 1 + 2 \times 5^3 \end{bmatrix}$$

$$= \begin{bmatrix} -127 & -84 \\ 378 & 251 \end{bmatrix},$$

$$A^5 = S\,\mathrm{diag}((-1)^5, 5^5)\,S^{-1} = \begin{bmatrix} -2 & 5^5 \\ 3 & -3 \times 5^5 \end{bmatrix} \begin{bmatrix} 1 & 1/3 \\ -1 & -2/3 \end{bmatrix} = \begin{bmatrix} -2 - 5^5 & -\frac{2(1+5^5)}{3} \\ 3(1+5^5) & 1 + 2 \times 5^5 \end{bmatrix}$$

$$= \begin{bmatrix} -3127 & -2084 \\ 9378 & 6251 \end{bmatrix}.$$