6.1

T/F.6. True. $L(y+u) = Ly + Lu = Ly = F$.

Prob.18. The functions $1, x^2, e^x, 0$ are all continuous functions on the whole real line. Therefore any initial value will give unique solution. Since $y(x) = 0$ does satisfy the equation and the initial condition, it must be THE solution. In other words, the only solution we can find is $y(x) = 0$.

Prob.20. Substituting $y(x) = e^{rx}$ into the equation we get

$$e^{rx}(r^2 - 2r - 3) = 0.$$

Solving this for r we have

$$r = -1, 3.$$

Therefore a general solution to the equation can be written as

$$y(x) = c_1 e^{-x} + c_2 e^{3x}.$$

Prob.24. Substituting $y(x) = e^{rx}$ into the equation we get

$$e^{rx}(r^3 - 3r^2 - r + 3) = e^{rx}(r-3)(r^2-1) = 0.$$

Solving this for r we have

$$r = \pm 1, 3.$$

Therefore a general solution to the equation can be written as

$$y(x) = c_1 e^{3x} + c_2 e^x + c_3 e^{-x}.$$

Prob.32. Substituting $y(x) = x^r$ into the equation we get

$$x^r[2r(r-1) + 5r + 1] = 0.$$

Solving this for r we have

$$r = -1, -1/2.$$

Therefore a general solution to the equation can be written as

$$y(x) = c_1 x^{-1} + c_2 x^{-1/2}.$$

Prob.34. Substituting $y(x) = x^r$ into the equation we get

$$x^r[r(r-1)(r-2) + 3r(r-1) - 6r] = x^r r(r^2 - 7) = 0.$$

Solving this for r we have

$$r = 0, \pm\sqrt{7}.$$

Therefore a general solution to the equation can be written as

$$y(x) = c_1 + c_2 x^{-\sqrt{7}} + c_3 x^{\sqrt{7}}.$$

6.2

Prob.6. The auxiliary equation is

$$r^2 - 6r + 9 = (r-3)^2 = 0.$$

Solving this for r we have

$$r = 3,$$

with multiplicity 2. Therefore a general solution to the equation can be written as

$$y(x) = c_1 e^{3x} + c_2 x e^{3x}.$$

Prob.12. The auxiliary equation is

$$r^2 - 2 = 0.$$

Solving this for r we have

$$r = \pm\sqrt{2}.$$

Therefore a general solution to the equation can be written as

$$y(x) = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}.$$

Prob.22. The auxiliary equation is

$$(r^2 + 3)(r + 1)^2 = 0.$$

Solving this for r we have

$$r = -1, \pm i\sqrt{3},$$

with 1 having multiplicity 2. The general solution to the equation can be written as

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 \sin(\sqrt{3}x) + c_4 \cos(\sqrt{3}x).$$

Prob.38. It's a constant coefficient equation, therefore we still use e^{rt} as our solution. The equation for r is

$$r^2 + 2cr + k^2 = 0.$$

When $c^2 < k^2$,

$$r = -c \pm \sqrt{c^2 - k^2} = -c \pm i\sqrt{k^2 - c^2}.$$

We have two distinct complex solutions for r. Therefore a general solution is(let $\omega = \sqrt{k^2 - c^2}$)

$$y(t) = c_1 e^{-ct}\cos(\omega t) + c_2 e^{-ct}\sin(\omega t).$$

Now we plug in the initial condition

$$y(0) = c_1 = y_0,$$

$$y'(0) = -cc_1 + \omega c_2 = 0.$$

The solution is

$$c_1 = y_0, \quad c_2 = cy_0/\omega.$$

Therefore

$$y(t) = \frac{y_0}{\omega} e^{-ct}[\omega\cos(\omega t) + c\sin(\omega t)].$$

Let ϕ be an angle *s.t.*

$$\sin\phi = \frac{\omega}{\sqrt{c^2+\omega^2}} = \frac{\omega}{k},$$

and

$$\cos\phi = \frac{c}{\sqrt{c^2+\omega^2}} = \frac{c}{k}.$$

Then using our favorite trig identity

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

we get

$$y(t) = \frac{y_0 k}{\omega} e^{-ct}\sin(\omega t + \phi).$$

Since we know $\sin\phi > 0, \cos\phi > 0$, $\phi \in (0, \pi/2)$, which means it's completely determined by its tan value. We can write without ambiguity that

$$\phi = \tan^{-1}\frac{\omega}{c}.$$

This is called damped harmonic oscillator. You can find more about this all over internet, including many nice pictures of it. For example
http://en.wikipedia.org/wiki/Harmonic_oscillator#Damped_harmonic_oscillator.

Prob.40. The solution will look like

$$c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

To make it 0 when $x \to \infty$, we need $r_1, r_2 < 0$, if both are real. If both are complex ($r_{1,2} = a \pm ib$), then we need $\Re(r_1), \Re(r_2) < 0$, which means $a < 0$.
If $a_1, a_2 > 0$, if r_1, r_2 are both real, we have $r_1 r_2 = a_2 > 0$, which means they have the same sign, and $r_1 + r_2 = -a_1 < 0$, which mean they are both negative. If

$r_{1,2} = a \pm ib$, then $2a = -a_1 < 0$, which means $a < 0$. From what we said above, both of these cases leads to $y \to 0$ when $x \to \infty$.
If $a_1 > 0, a_2 = 0$, we have $r_1 = -a_1$ and $r_2 = 0$, and the solution now looks like

$$y(x) = c_1 e^{-a_1 x} + c_2.$$

Since $a_1 > 0$,

$$\lim_{x\to\infty} y(x) = c_2.$$

If $a_1 = 0, a_2 > 0$, we have $r_{1,2} = \pm i\sqrt{a_2}$. the solution now looks like

$$y(x) = c_1 \cos(\sqrt{a_2}x) + c_2 \sin(\sqrt{a_2}x),$$

which is bounded.

6.3

Prob.18. We first obtain the general solution to

$$y'' + y = (D^2 + 1)y(x) = 0$$

as

$$y_c(x) = c_1 \cos x + c_2 \sin x.$$

Since

$$(D - 1)6e^x = 0,$$

the trial solution $y_p(x)$ satisfies

$$(D - 1)(D^2 + 1)y_p(x) = 0,$$

which has general solution as

$$c_1 \cos x + c_2 \sin x + c_3 e^x.$$

Since the first two terms are already covered in $y_c(x)$, we might as well set it to 0 and pick

$$y_p(x) = c_3 e^x.$$

Plug this in the original equation and we get

$$2c_3 e^x = 6e^x.$$

Therefore

$$c_3 = 3.$$

And the final answer is

$$y(x) = c_1 \cos x + c_2 \sin x + 3e^x.$$

Prob.20. We first obtain the general solution to

$$y'' + 16y = (D^2 + 16)y(x) = 0$$

as

$$y_c(x) = c_1 \cos(4x) + c_2 \sin(4x).$$

Since

$$(D^2 + 1)4\cos(x) = 0,$$

the trial solution $y_p(x)$ satisfies

$$(D^2 + 1)(D^2 + 16)y_p(x) = 0,$$

which has general solution as

$$c_1 \cos(4x) + c_2 \sin(4x) + c_3 \cos(x) + c_4 \sin(x).$$

Since the first two terms are already covered in $y_c(x)$, we might as well set it to 0 and pick

$$y_p(x) = c_3 \cos(x) + c_4 \sin(x).$$

Plug this in the original equation and we get

$$15[c_3 \cos(x) + c_4 \sin(x)] = 4\cos(x).$$

Therefore

$$c_3 = \frac{4}{15}, \quad c_4 = 0.$$

And the final answer is

$$y(x) = c_1 \cos(4x) + c_2 \sin(4x) + \frac{4}{15}\cos(x).$$

Prob.26. We first obtain the general solution to

$$y''' - y'' + y' - y = (D^2 + 1)(D - 1)y(x) = 0$$

as

$$y_c(x) = c_1 e^x + c_2 \cos(x) + c_3 \sin(x).$$

Since

$$(D + 1)9e^{-x} = 0,$$

the trial solution $y_p(x)$ satisfies

$$(D^2 + 1)(D + 1)(D - 1)y_p(x) = 0,$$

which has general solution as

$$c_1 e^x + c_2 \cos(x) + c_3 \sin(x) + c_4 e^{-x}.$$

Since the first three terms are already covered in $y_c(x)$, we might as well set it to 0 and pick

$$y_p(x) = c_4 e^{-x}.$$

Plug this in the original equation and we get

$$-4c_4 e^{-x} = 9e^{-x}.$$

Therefore

$$c_4 = -9/4.$$

And the final answer is

$$y(x) = c_1 e^x + c_2 \cos(x) + c_3 \sin(x) - \frac{9}{4}e^{-x}.$$