Math 104 – Calculus

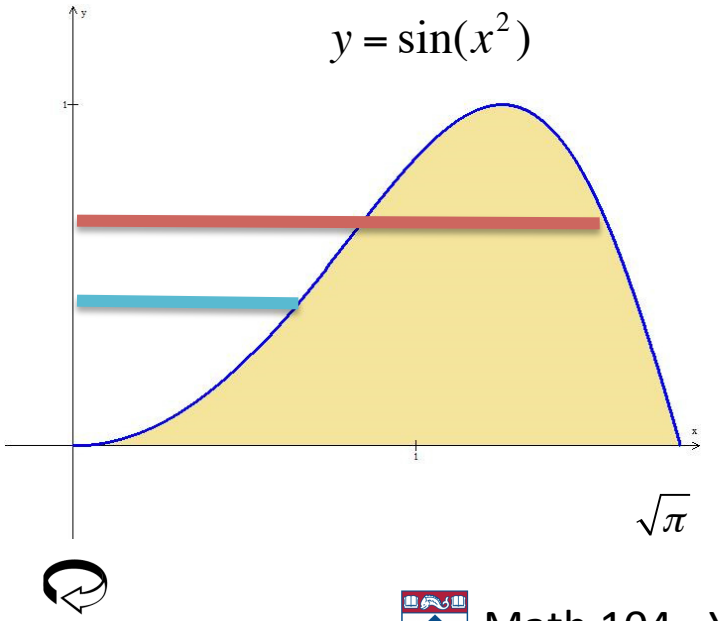
6.2 Volume by Cylindrical Shells

Volumes by Cylindrical Shells

- Sometimes finding the volume of a solid of revolution is impossible by the disk/washer method.

E.g., rotate the region between the curve $y = \sin(x^2)$ and the x-axis about the y-axis.

- The outer and inner radius use the same curve
- Hard to write the radii in terms of y



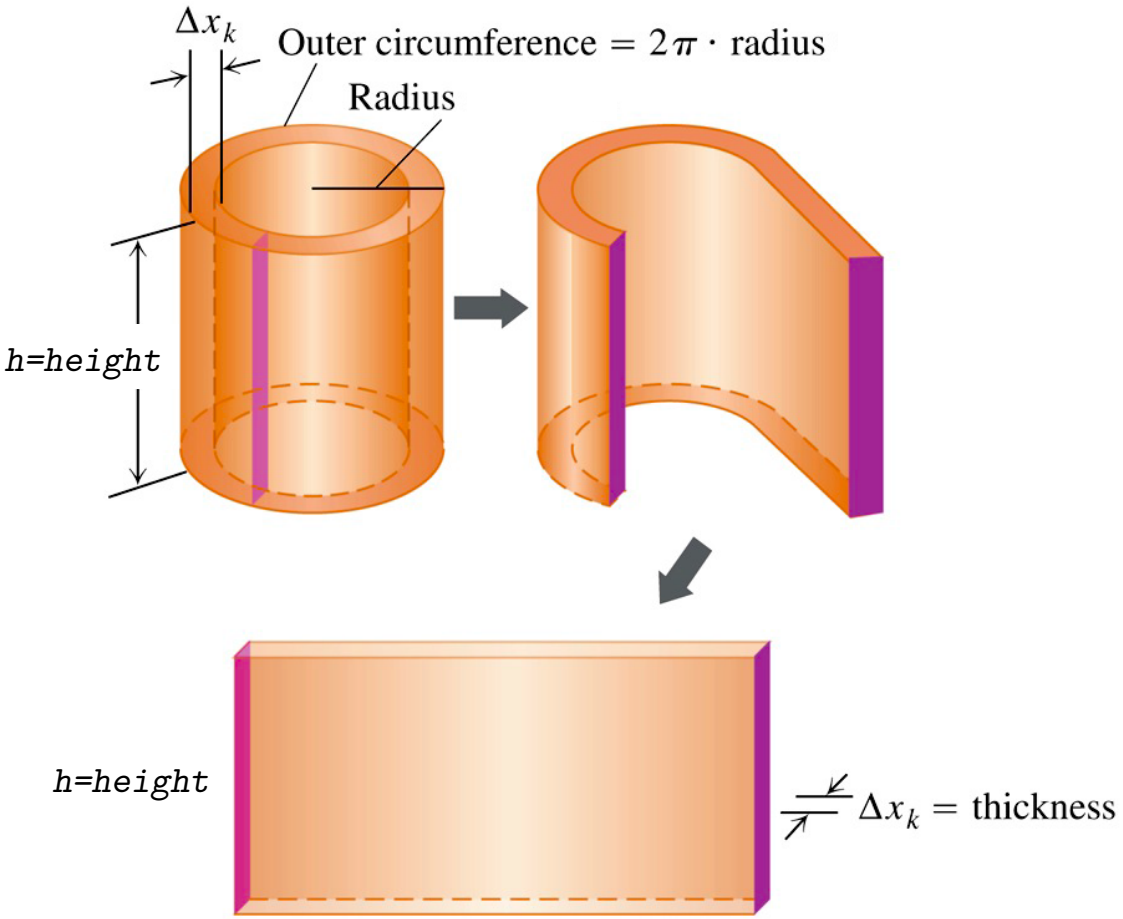
Volume by Cylindrical Shells

- In these cases, we partition the 2D region with rectangles parallel to the rotation axis, which rotate into cylindrical shells.
- Animation: http://www.mathdemos.org/mathdemos/shellmethod/sinshells.gif

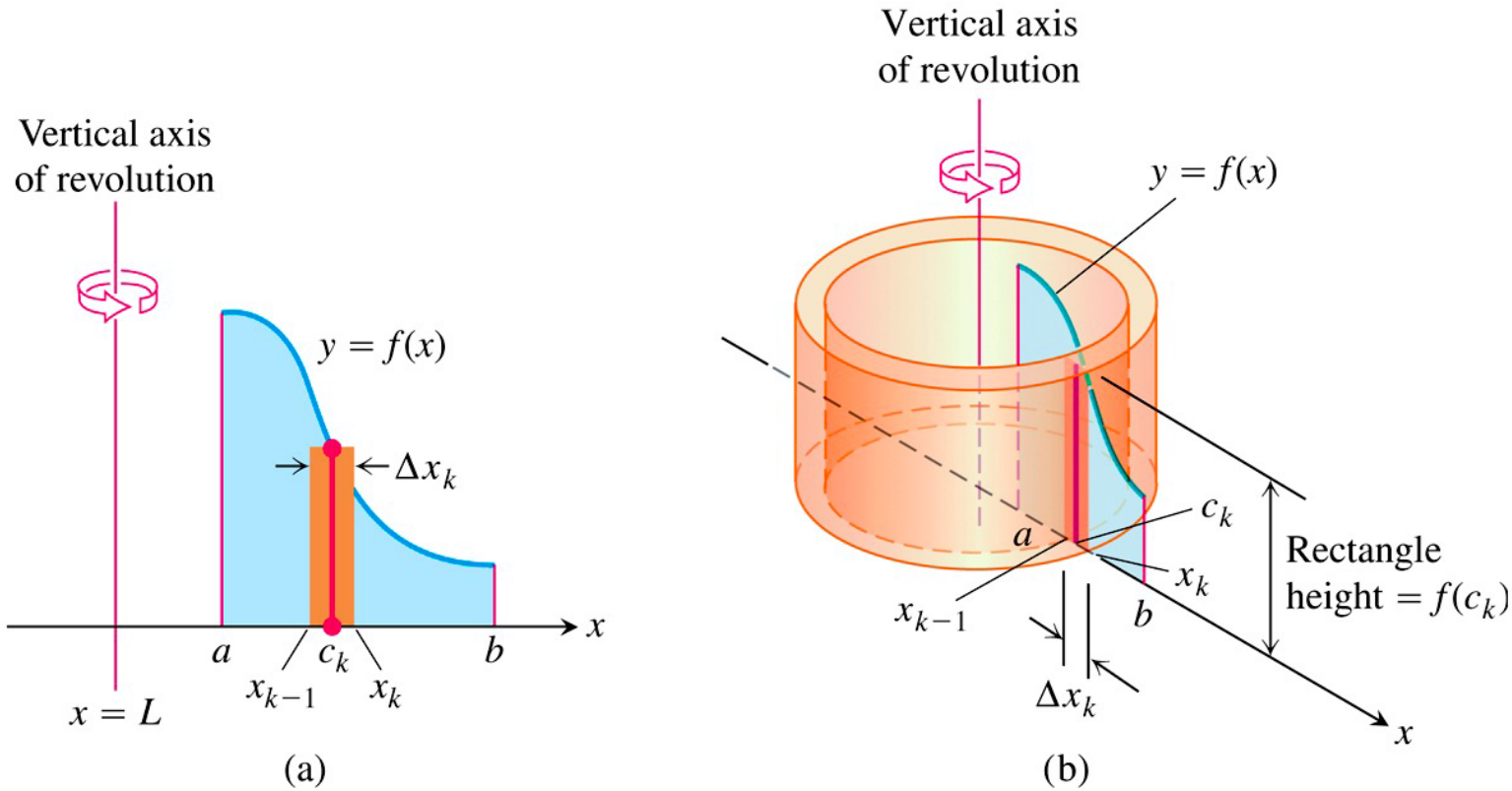
Volume of a Cylindrical Shell

- The volume of a cylindrical shell can be computed by cutting and "unrolling".

$$\begin{aligned}\Delta V &= \text{perimeter} \times \text{height} \\ &\quad \times \text{thickness} \\ &= 2\pi \times \text{radius} \times \text{height} \\ &\quad \times \text{thickness}\end{aligned}$$



Riemann Sum Approximation

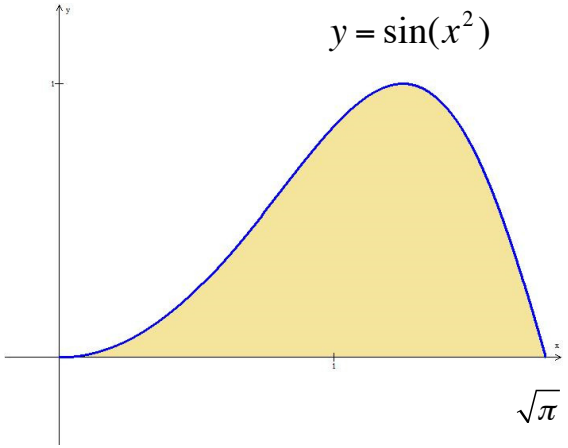


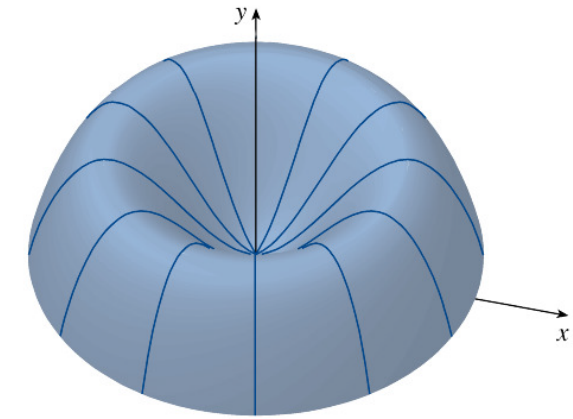
Riemann sum converges to an integral:

$$\sum_k \Delta V_k = \sum_k 2\pi(x_k - L)f(c_k)\Delta x_k \xrightarrow{\Delta x_k \to 0} \int_a^b 2\pi(x - L)f(x)dx$$

Example

Rotate the region between the curve $y = \sin(x^2)$ and the x-axis about the y-axis. Find the volume





Shell Method

- In general,

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\, dx$$

Distance from axis of rotation to a typical rectangle

Height of a typical rectangle

For a vertical axis of rotation; When the axis is horizontal, use ***dy***

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. *Find the limits of integration* for the thickness variable.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

Examples

1. The region in the first quadrant bounded by $x = 0$, $y = 1$ and $y = 5 - x^2$ is revolved around the y-axis. Find the volume.

2. Calculate the volume of revolving the region bounded by $y = 4 - x^2$, $y = 0$ around the line $y = 4$.

Comparison of Methods

Method	Typical Rectangle	Vertical Axis	Horizontal Axis
Disk/Washer	Perpendicular to axis	Use dy and x=g(y)	Use dx and y=f(x)
Shell	Parallel to axis	Use dx and y=f(x)	Use dy and x=g(y)

To shell or not to shell?

- Sometimes it is easier to use one method over the other. Sometimes the problem is unsolvable using one method.

3. Find the volume of the solid obtain by revolving the region in the first quadrant given by $y = \dfrac{\sin x}{x}$ about the y-axis.

Examples (cont.)

4. Find the volume of the solid obtained by revolving the region in the first quadrant bounded by $y = x + 2$, $y = x^2$ and the y-axis about the y-axis.

5. Revolve the region below $y = -3x^4 + 3x$ (in the first quadrant) about the x-axis and calculate the volume.

Animations

- Shell Method: http://www.mathdemos.org/mathdemos/washermethod/gallery/gallery.html
- Washer Method: http://www.mathdemos.org/mathdemos/washermethod/gallery/gallery.html