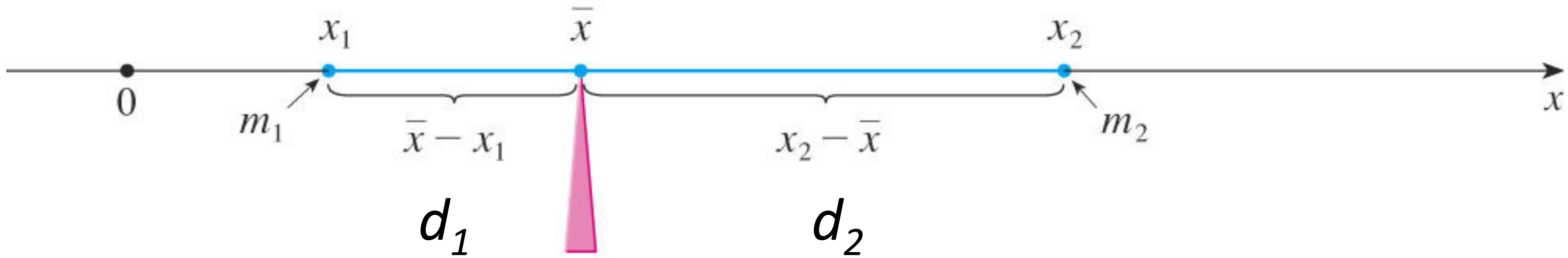
Math 104 – Calculus

6.6 Moments and Centers of Mass

Balancing Masses



- Along the line, mass m_1 at point x_1, mass m_2 at x_2.
- Archimedes' Law of Lever: rod will be balanced if

$$m_1 d_1 = m_2 d_2$$

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1\bar{x} - m_1 x_1 = m_2 x_2 - m_2\bar{x}$$

$$(m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2$$

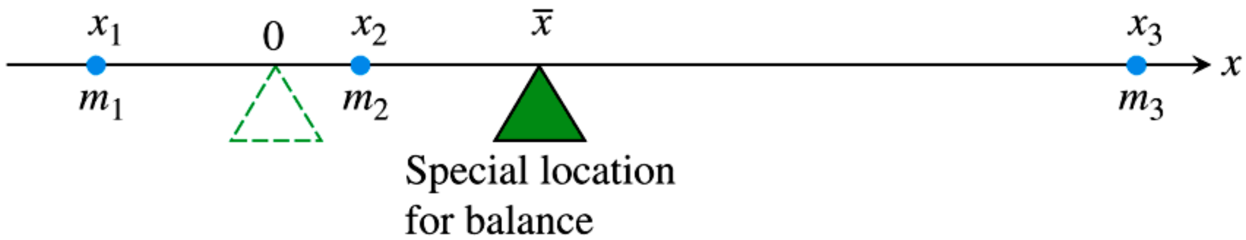
Center of Mass:

Moment of the system about the origin

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Total Mass

Center of Mass Along a Line



- A system of n masses: $m_1, m_2, \ldots, m_n$
- Center of mass:

$$\bar{x} = \frac{m_1 x_1 + \cdots + m_n x_n}{m_1 + \cdots + m_n}$$

- Moment of the system about the origin: measure the tendency of the system to rotate about the origin.

$$M_0 = m_1 x_1 + \cdots + m_n x_n$$

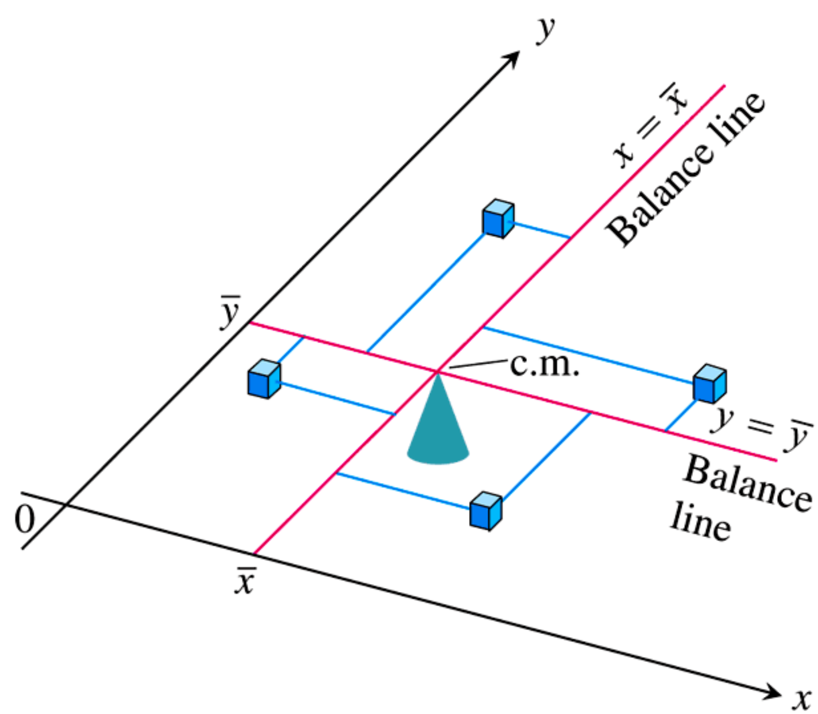
Balancing on the Plane

- For a planar region, it can rotate about either x or y-axis. So there are two moments:
  - Moment about x-axis: $M_x = m_1 y_1 + \cdots m_n y_n$.
  - Moment about y-axis: $M_y = m_1 x_1 + \cdots m_n x_n$.
- Total Mass:

  $$M = m_1 + \cdots + m_n$$

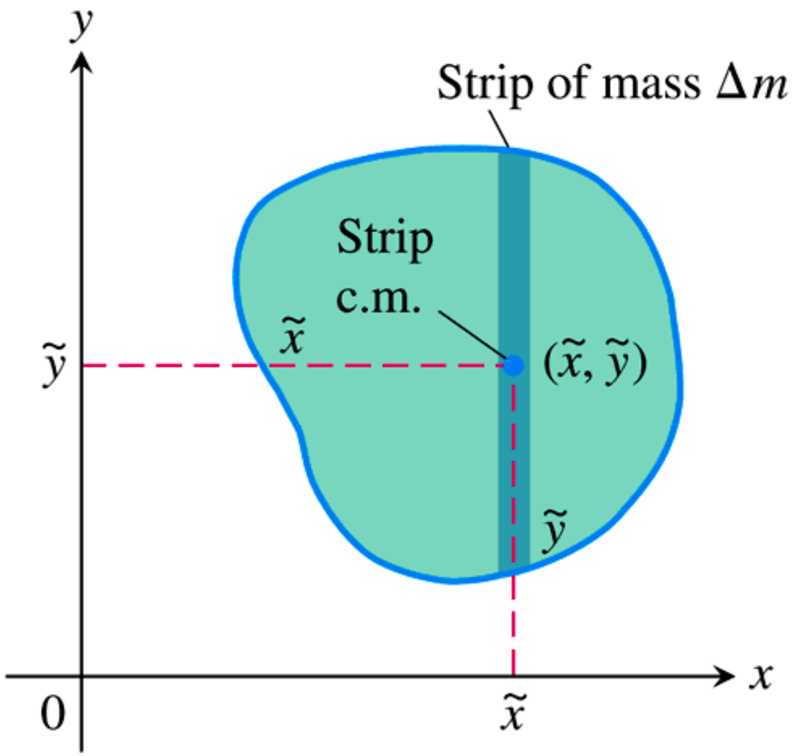
- The center of mass is:

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$



Center of Mass of a 2D Region

- To compute moments and total mass of a region with a given density (mass to area ratio), we partition it into strips and do a Riemann sum approximation.



Center of Mass of a 2D Region

Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the *xy*-Plane

Moment about the x-axis: $$M_x = \int \tilde{y}\, dm$$

Moment about the y-axis: $$M_y = \int \tilde{x}\, dm$$

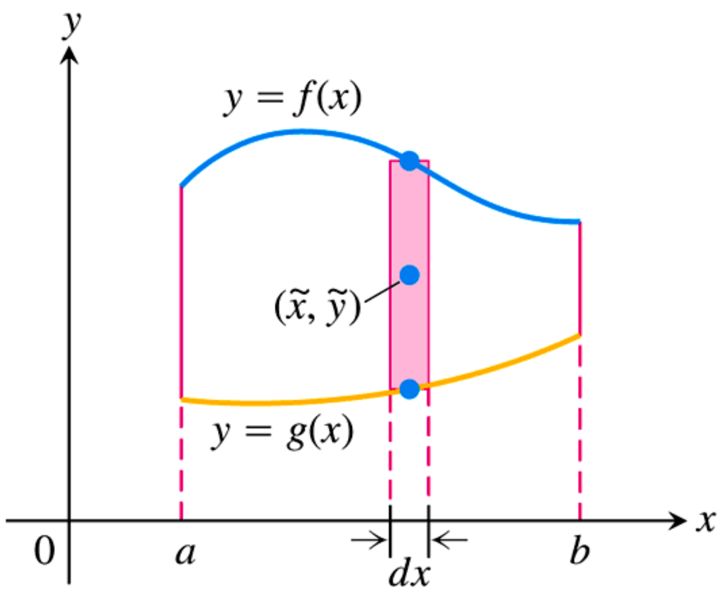
Mass: $$M = \int dm \qquad (5)$$

Center of mass: $$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$

Center of Mass of a 2D Region

- If the density only depends on the x coordinate, total mass is given by

$$M = \int_a^b \delta(x)[f(x) - g(x)]dx$$



- The moments are:

$$M_y = \int_a^b x\delta(x)[f(x) - g(x)]dx$$

$$M_x = \int_a^b \frac{1}{2}[f(x) + g(x)]\delta(x)[f(x) - g(x)]dx$$

$$= \frac{1}{2}\int_a^b \delta(x)[f(x)^2 - g(x)^2]dx$$

$$\boxed{\begin{aligned} \tilde{x} &= x \\ \tilde{y} &= \frac{f(x) + g(x)}{2} \\ \tilde{dm} &= \delta dA \\ &= \delta(x)[f(x) - g(x)]dx \end{aligned}}$$

Centroid

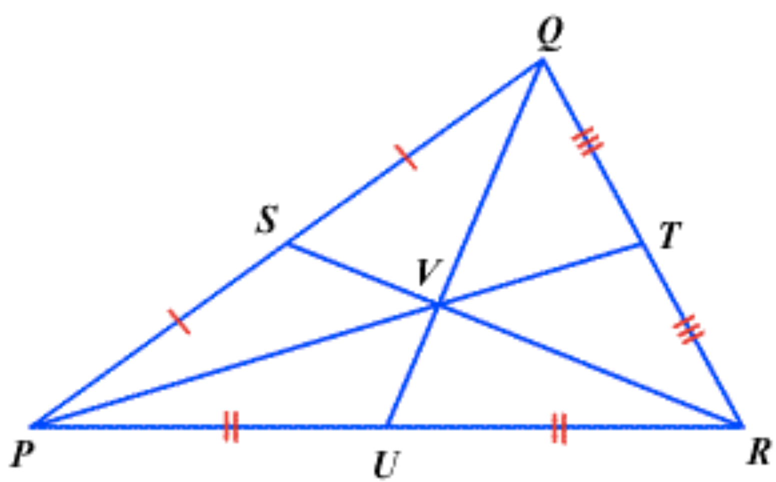
- If the density is constant the formula simplify:

$$\bar{x} = \frac{M_y}{M} = \frac{\int_a^b x[f(x) - g(x)]dx}{\boxed{\int_a^b [f(x) - g(x)]dx}}$$

Total area of the region *A*

$$\bar{y} = \frac{M_x}{M} = \frac{\int_a^b [f(x)^2 - g(x)^2]dx}{2\int_a^b [f(x) - g(x)]dx}$$

- In this case the value of the density is irrelevant. We also call the center of mass the *centroid* of the region.



Examples

1. Find the center of mass of a thin plate between the x-axis and $y = 2/x^2$, $1 \le x \le 2$, if the density is $\delta(x) = x^2$.

2. Find the centroid of an isosceles triangle whose base is on the x-axis, $-1 \le x \le 1$ and whose height is 3.

Examples

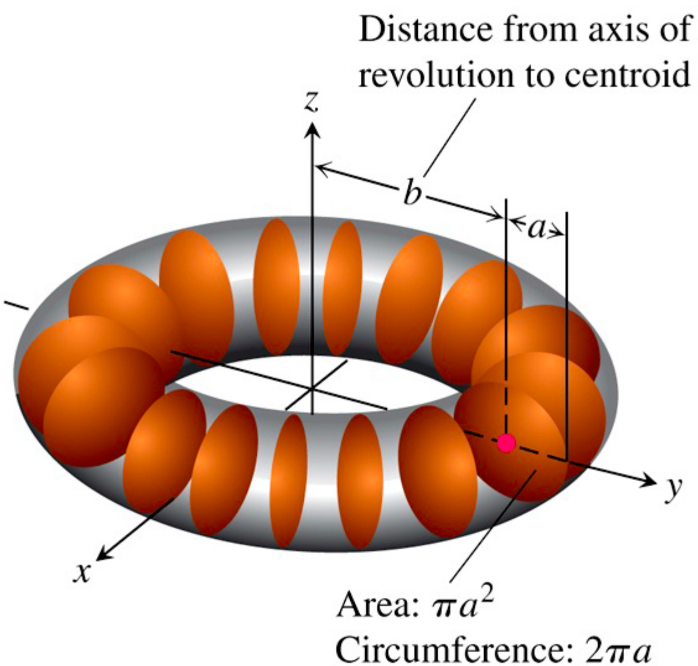
3. Find the center of mass of a plate of constant density given by the region between $y = x - x^2$ and $y = -x$.

Pappus' Theorem

THEOREM 1 Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$V = 2\pi\rho A. \tag{9}$$



Example

- Find the centroid of a upper-half disk of radius a using Pappus' Theorem.