

DOLBEAULT DGA OF FORMAL NEIGHBORHOODS AND L_∞ -ALGEBROIDS

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ABSTRACT.

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1. INTRODUCTION

In § 2 we define a Dolbeault dga $\mathcal{A}^\bullet(X_Y^{(\infty)})$ for the formal neighborhood $X_Y^{(\infty)}$ of an arbitrary closed holomorphic embedding $i : X \hookrightarrow Y$ of complex manifolds. We show that its sheafy version gives a fine resolution of the structure sheaf of holomorphic functions over $X_Y^{(\infty)}$.

Key words and phrases. ...

In § 3 we review basics of formal geometry and Kapranov's construction in [?], which, in our language, gives a concrete description of the Dolbeault dga of the formal neighborhood of the diagonal embedding $\Delta : X \hookrightarrow X \times X$ using Atiyah class. We reprove the theorem, however, from a different point of view. Namely, instead of working on the holomorphic exponential map and using the language of formal vector fields, we take the 'dual' picture and look at how functions on the formal neighborhood $X_{X \times X}^{(\infty)}$ of the diagonal are pullbacked to functions on the formal neighborhood of the zero section of the tangent bundle TX . It turns out to be the same as taking 'Taylor series' using jets of flat and torsion-free $(1, 0)$ -connections. An alternative proof of Kapranov's theorem is obtained via this approach.

In § 4 we generalize Kapranov's theorem to the general case of arbitrary embedding $i : X \hookrightarrow Y$, in which many other differential geometric quantities other than the curvature, such as Kodaira-Spencer class and shape operator, come into the picture. For convenience we only discuss the Kähler case. We construct an isomorphism from our canonical yet abstractly defined Dolbeault dga $\mathcal{A}^\bullet(X_Y^{(\infty)})$ to a concrete dga, namely the completed symmetric algebra $\mathcal{A}^{0,\bullet}(\widehat{S}(N_{X/Y}^\vee))$ of the conormal bundle of X in Y , and compute the differential on it. The main result is Theorem 4.6 for the final answer.

§ 5 is recently added for readers who prefer the language of Lie algebroids. We just repackage the results in 4 in terms of an L_∞ -algebroid structure on the Dolbeault complex $\mathcal{A}^{\bullet-1}(N_{X/Y})$ of the normal bundle. It will be shown that this L_∞ -algebroid governs the embedded deformation s of the complex submanifold X inside Y . This part is still under construction.

2. DOLBEAULT DGA OF FORMAL NEIGHBORHOODS

2.1. Definition: a geometric approach. Let Y be a complex manifold, X a closed complex submanifold of X via embedding $i = i_{X/Y} : X \hookrightarrow Y$, which is defined by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$. For each $r \geq 0$, we can define the r -th formal neighborhood $(X_Y^{(r)}, \mathcal{O}_{X_Y^{(r)}})$ with X the underlying topological space and $\mathcal{O}_{X_Y^{(r)}} = \mathcal{O}_Y/\mathcal{I}^{r+1}$ the structure sheaf. All these sheaves form an inverse

system in an obvious way, and by passing to the inverse limit we get the (complete) formal neighborhood $X_Y^{(\infty)}$ with structure sheaf

$$\mathcal{O}_{X_Y^{(\infty)}} = \varprojlim_r \mathcal{O}_{X_Y^{(r)}} = \varprojlim_r \mathcal{O}_X / \mathcal{I}^{r+1}$$

Roughly, the sheaf $\mathcal{O}_X / \mathcal{I}^{r+1}$ records holomorphic functions on Y defined around X up to r -th order. We adopt the same idea for smooth functions. Denote by $\mathcal{A}^{0,\bullet}(Y) = \mathcal{A}^{0,\bullet}(Y)$ the Dolbeault complex of Y , which is a differential graded algebra (dga) with the differential being the usual $(0, 1)$ -part of the De Rham differential

$$\bar{\partial} : \mathcal{A}^{0,\bullet}(Y) \rightarrow \mathcal{A}^{0,\bullet+1}(Y).$$

Moreover, it admits a natural Fréchet algebra structure, with respect to which $\bar{\partial}$ is continuous. We call such dga as a *Fréchet dga*. For each natural number r , we define

$$\mathfrak{a}_r^k = \mathfrak{a}_r^k(X/Y\text{nn}) := \left\{ \omega \in \mathcal{A}^{0,k}(Y) \mid \begin{array}{l} i^*(\mathcal{L}_{V_1} \mathcal{L}_{V_2} \cdots \mathcal{L}_{V_l} \omega) = 0, \forall V_j \in C^\infty(T^{1,0}Y), \\ \forall 1 \leq j \leq l, 0 \leq l \leq r. \end{array} \right\} \quad (2.1)$$

a subset of each $\mathcal{A}^{0,k}(Y)$, where $T^{1,0}Y$ is the $(1, 0)$ -part of the complexified tangent bundle of Y and \mathcal{L}_V denote the usual Lie derivative.

Proposition 2.1. \mathfrak{a}_r^k is a closed dg-ideal of $\mathcal{A}^{0,\bullet}(Y)$ with respect to the Fréchet structure. In particular, it is closed under the action of $\bar{\partial}$.

Before proving the proposition, we need some preliminaries on a dg-version of the Cartan calculus in our context. First notice that, if V is a $(1, 0)$ -vector field and ω is a $(0, k)$ -form over Y , then the Lie derivative $\mathcal{L}_V \omega$ is still a $(0, k)$ -form. Moreover, this operation is $C^\infty(Y)$ -linear with respect to V , i.e.,

$$\mathcal{L}_{gV} \omega = g \cdot \mathcal{L}_V \omega, \quad \forall g \in C^\infty(Y). \quad (2.2)$$

Indeed, by Cartan's formula,

$$\mathcal{L}_V \omega = \iota_V d\omega + d\iota_V \omega.$$

But $\iota_V \omega = 0$ since we are contracting a $(1, 0)$ -vector field with a $(0, k)$ -form. So we have

$$\mathcal{L}_V \omega = \iota_V d\omega = \iota_V (\partial\omega + \bar{\partial}\omega) = \iota_V \partial\omega,$$

which is obviously linear in V . Thus we extend the Lie derivative into an operator

$$\mathcal{L}_{(\cdot)}(\cdot) : \mathcal{A}^{0,k}(T^{1,0}Y) \times \mathcal{A}^{0,l}(Y) \rightarrow \mathcal{A}^{0,k+l}(Y), \quad \forall k, l \geq 0$$

by defining

$$\mathcal{L}_{\eta \otimes V} \zeta = \iota_{\eta \otimes V} \partial \zeta = \eta \wedge \mathcal{L}_V \zeta$$

for any $\eta, \zeta \in \mathcal{A}^{0,\bullet}(Y)$ and $V \in C^\infty(T^{1,0}Y)$. One can check that, in this way, we obtain a homomorphism of dg-Lie algebras

$$\theta : \mathcal{A}^{0,\bullet}(T^{1,0}Y) \rightarrow \text{Der}_{\mathbb{C}}^{\bullet}(\mathcal{A}^{0,\bullet}(Y), \mathcal{A}^{0,\bullet}(Y)), \quad \eta \otimes V \mapsto \mathcal{L}_{\eta \otimes V},$$

which is also $\mathcal{A}^{0,\bullet}(Y)$ -linear. In particular, we have

$$\mathcal{L}_{\bar{\partial}V} \omega = \bar{\partial} \mathcal{L}_V \omega - \mathcal{L}_V \bar{\partial} \omega, \quad (2.3)$$

for any $(1, 0)$ -vector field V and $(0, k)$ -form ω over Y .

Proof of Proposition 2.1. It is obvious that \mathfrak{a}_r^k is a closed graded ideal of $\mathcal{A}^{0,\bullet}(Y)$, so we only prove it is closed under $\bar{\partial}$. Observe that, by the linearity of \mathcal{L} (2.2), if we substitute in the definition (2.1) those Lie derivatives \mathcal{L}_{V_i} by $\mathcal{L}_{\mathcal{V}_i}$ for any $\mathcal{V}_i \in \mathcal{A}_Y^{0,\bullet}(T^{1,0}Y)$, nothing will be changed and we get an equivalent definition of \mathfrak{a}_r . By (2.3), the commutator

$$[\bar{\partial}, \mathcal{L}_{V_1} \cdots \mathcal{L}_{V_l}] = \sum_{i=1}^l \mathcal{L}_{V_1} \cdots \mathcal{L}_{\bar{\partial}V_i} \cdots \mathcal{L}_{V_l}$$

is still a differential operator on $\mathcal{A}_Y^{0,\bullet}$ of order $\leq l$. Thus if $\omega \in \mathfrak{a}_r^{\bullet}$, then

$$i^* \mathcal{L}_{V_1} \cdots \mathcal{L}_{V_l} \bar{\partial} \omega = \bar{\partial} i^* \mathcal{L}_{V_1} \cdots \mathcal{L}_{V_l} \omega - \sum_{i=1}^l i^* \mathcal{L}_{V_1} \cdots \mathcal{L}_{\bar{\partial}V_i} \cdots \mathcal{L}_{V_l} \omega = 0,$$

for any $0 \leq l \leq r$, which means that $\bar{\partial} \omega$ also lies in \mathfrak{a}_r^{\bullet} . Hence \mathfrak{a}_r^{\bullet} is a dg-ideal. \square

Definition 2.2. The *Dolbeault dga of the r -th order formal neighborhood of X in Y* (or *Dolbeault dga of $X_Y^{(r)}$*) is the quotient dga

$$\mathcal{A}^\bullet(X_Y^{(r)}) := \mathcal{A}^{0,\bullet}(Y) / \mathfrak{a}_r^\bullet.$$

Note that $\mathcal{A}^\bullet(X^{(0)}) = \mathcal{A}^{0,\bullet}(X)$. Moreover, we have a filtration of dg-ideals

$$\mathfrak{a}_0^\bullet \supset \mathfrak{a}_1^\bullet \supset \mathfrak{a}_2^\bullet \supset \cdots$$

which induces an inverse system of dgas

$$\mathcal{A}^{0,\bullet}(X) = \mathcal{A}^\bullet(X_Y^{(0)}) \leftarrow \mathcal{A}^\bullet(X_Y^{(1)}) \leftarrow \mathcal{A}^\bullet(X_Y^{(2)}) \leftarrow \cdots,$$

we can also define the *Dolbeault dga of the infinite-order formal neighborhood of X in Y* (or *Dolbeault dga of $X_Y^{(\infty)}$*) as the inverse limit

$$\mathcal{A}^\bullet(X_Y^{(\infty)}) := \varprojlim_r \mathcal{A}^\bullet(X_Y^{(r)}).$$

We endow $\mathcal{A}^\bullet(X_Y^{(\infty)})$ with the initial topology which makes it into a Fréchet dga.

Suppose there are two closed embeddings of complex manifolds $X \hookrightarrow Y$ and $X' \hookrightarrow Y'$, together with a holomorphic map $f : Y \rightarrow Y'$ which maps X into X' . Then obviously the pullback morphism $f^* : \mathcal{A}^{0,\bullet}(Y') \rightarrow \mathcal{A}^{0,\bullet}(Y)$ maps the dg-ideals $\mathfrak{a}_r^\bullet(X'/Y')$ into $\mathfrak{a}_r^\bullet(X/Y)$, hence induces homomorphisms of dgas

$$f^* : \mathcal{A}^\bullet(X_{Y'}^{(r)}) \rightarrow \mathcal{A}^\bullet(X_Y^{(r)})$$

and

$$f^* : \mathcal{A}^\bullet(X_{Y'}^{(\infty)}) \rightarrow \mathcal{A}^\bullet(X_Y^{(\infty)}).$$

Moreover, if $X'' \hookrightarrow Y''$ is a third embedding and $g : Y' \rightarrow Y''$ maps X' into X'' , then

$$f^* \circ g^* = (g \circ f)^* : \mathcal{A}^\bullet(X_{Y''}^{(r)}) \rightarrow \mathcal{A}^\bullet(X_Y^{(r)}),$$

where r is any nonnegative integer or ∞ . In other words, for each r the association of Dolbeault dgas gives a functor from the category of closed embeddings of complex manifolds to the category of (Fréchet) dgas.

If a closed embedding $X \rightarrow Y'$ factors through another one $X \rightarrow Y$ where Y is an open submanifold of Y' , the restriction map $\mathcal{A}^{0,\bullet}(Y') \rightarrow \mathcal{A}^{0,\bullet}(Y)$ actually induces a natural isomorphism $\mathcal{A}^\bullet(X_{Y'}^{(r)}) \cong \mathcal{A}^\bullet(X_Y^{(r)})$, since for any smooth function f defined on Y , one can always find another function f' on Y' such that f and f' coincide on some open neighborhood of X .

Inspired by this, we associate to each open subset U of X the dga $\mathcal{A}^\bullet(U_V^{(r)})$, where V is an arbitrary open subset of the ambient manifold Y such that $X \cap V = U$. One can check that this gives a sheaf of dgas

There are sheafy versions of the constructions above. Denote by $\tilde{\mathcal{A}}_Y^{0,\bullet}$ the sheaf of dgas of $(0, q)$ -forms with the Dolbeault differential $\bar{\partial}$, i.e., for any open subset $U \subset Y$,

$$\tilde{\mathcal{A}}_Y^{0,\bullet}(U) = \mathcal{A}^{0,\bullet}(U).$$

The complex of sheaves $(\tilde{\mathcal{A}}^{0,\bullet}(Y), \bar{\partial})$ is a fine resolution of \mathcal{O}_Y . Inside $\tilde{\mathcal{A}}^{0,\bullet}(Y)$ there is a subsheaf $\tilde{\mathfrak{a}}_r^\bullet = \tilde{\mathfrak{a}}_{X/Y,r}^\bullet$ of dg-ideals of $\tilde{\mathcal{A}}^{0,\bullet}(Y)$, whose sections over any open subset $U \subset Y$ are

$$\tilde{\mathfrak{a}}_r^\bullet(U) = \mathfrak{a}_r^\bullet(X \cap U/U).$$

It is also a fine sheaf and its global sections over Y are just \mathfrak{a}_r^\bullet . We define the *sheaf of Dolbeault dgas of $X_Y^{(r)}$* as the quotient sheaf of dgas

$$\tilde{\mathcal{A}}_{X_Y^{(r)}}^\bullet = \tilde{\mathcal{A}}_Y^{0,\bullet} / \tilde{\mathfrak{a}}_r^\bullet$$

with the differential $\bar{\partial}$ inherited from that of $\tilde{\mathcal{A}}^{0,\bullet}(Y)$. Since $(\tilde{\mathfrak{a}}_r^\bullet)_x = \tilde{\mathcal{A}}^{0,\bullet}(Y)_x$ for $x \notin X$, $\tilde{\mathcal{A}}_{X_Y^{(r)}}^\bullet$ has support on X , so we can think of $\tilde{\mathcal{A}}_{X_Y^{(r)}}^\bullet$ as a sheaf over X . Then over any open subset $V \subset X$, sections of $\tilde{\mathcal{A}}_{X_Y^{(r)}}^\bullet$ are just

$$\tilde{\mathcal{A}}_{X_Y^{(r)}}^\bullet(V) = \tilde{\mathcal{A}}_Y^{0,\bullet}(U) / \tilde{\mathfrak{a}}_{X/Y,r}^\bullet(U) = \mathcal{A}^{0,\bullet}(U) / \mathfrak{a}_r^\bullet(V/U) = \mathcal{A}^\bullet(V_U^{(r)}),$$

where U is an arbitrary open subset of Y such that $U \cap X = V$ and $\mathcal{A}^\bullet(V_U^{(r)})$ does not depend on the choice of U as mentioned before. In particular, global sections of $\tilde{\mathcal{A}}_{X_Y^{(r)}}^\bullet$ over X are just the Dolbeault dga $\mathcal{A}^\bullet(X_Y^{(r)})$ we have defined.

Moreover, there are natural inclusions of sheaves $\mathcal{I}^{r+1} \hookrightarrow \tilde{\mathfrak{a}}_r^0$ which gives a cochain complex of sheaves

$$0 \rightarrow \mathcal{I}^{r+1} \rightarrow \tilde{\mathfrak{a}}_r^0 \xrightarrow{\bar{\partial}} \tilde{\mathfrak{a}}_r^1 \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \tilde{\mathfrak{a}}_r^n \rightarrow 0,$$

which is a subcomplex of

$$0 \rightarrow \mathcal{O}_Y \rightarrow \tilde{\mathcal{A}}_Y^{0,0} \xrightarrow{\bar{\partial}} \tilde{\mathcal{A}}_Y^{0,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \tilde{\mathcal{A}}_Y^{0,n} \rightarrow 0$$

where $n = \dim_{\mathbb{C}} Y$. The complex of quotients is

$$0 \rightarrow \mathcal{O}_{X_Y^{(r)}} \rightarrow \tilde{\mathcal{A}}_{X_Y^{(r)}}^0 \xrightarrow{\bar{\partial}} \tilde{\mathcal{A}}_{X_Y^{(r)}}^1 \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \tilde{\mathcal{A}}_{X_Y^{(r)}}^n \rightarrow 0.$$

We will see in the following example and proposition that this complex actually ends up at the term $\tilde{\mathcal{A}}_{X_Y^{(r)}}^m$ where m is the dimension of X , and it is also exact.

Example 2.3. Let $Y = \mathbb{C}^{n+m} = \mathbb{C}^n \times \mathbb{C}^m$ with linear coordinates (z_1, \dots, z_{n+m}) and $X = 0 \times \mathbb{C}^m$ be the linear subspace of Y defined by equations $z_1 = \dots = z_n = 0$. As usual $i : X \hookrightarrow Y$ denotes the embedding. We have a homomorphism of dgas

$$\begin{aligned} \mathcal{T}_r : (\mathcal{A}^{0,\bullet}(Y), \bar{\partial}) &\rightarrow (\mathcal{A}^{0,\bullet}(X) \otimes_{\mathbb{C}} \mathbb{C}[z_1, \dots, z_n] / (z_1, \dots, z_n)^{r+1}, \bar{\partial} \otimes 1), \\ \omega &\mapsto \sum_{|\alpha| \leq r} \frac{1}{\alpha!} i^*(\partial^\alpha \omega) \otimes z^\alpha \end{aligned} \quad (2.4)$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ are multi-indices of nonnegative integers, $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}$ and $\partial_j = \mathcal{L}_{\partial/\partial z_j}$ and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$. Obviously \mathcal{T}_r is surjective. Moreover, the kernel of \mathcal{T} is exactly \mathfrak{a}_r^\bullet . To see this, observe that in the definition (??) of \mathfrak{a}_r^\bullet , we only need to take those test vector fields V_i in the transversal direction to $X = 0 \times \mathbb{C}^m$ and they can even be the constant vector fields $\partial/\partial z_i$ for the same reason as in ???. Thus we have an isomorphism of dgas

$$\tilde{\mathcal{T}}_r : \mathcal{A}^\bullet(X_Y^{(r)}) \xrightarrow{\cong} \mathcal{A}^{0,\bullet}(X) \otimes_{\mathbb{C}} \mathbb{C}[z_1, \dots, z_n] / (z_1, \dots, z_n)^{r+1} \quad (2.5)$$

Take inverse limits of the domains and codomains of $\tilde{\mathcal{T}}_r$ and note that $\tilde{\mathcal{T}}_r$ are compatible with the two inverse systems, we hence obtain an isomorphism

between Fréchet dgas

$$\tilde{\mathcal{T}}_\infty : \mathcal{A}^\bullet(X_Y^{(\infty)}) \xrightarrow{\cong} \mathcal{A}^{0,\bullet}(X)[[z_1, \dots, z_n]].$$

Here we endow $\mathcal{A}^{0,\bullet}(X)[[z_1, \dots, z_n]]$ with the initial Fréchet algebra structure. In particular, we see that the Dolbeault dgas actually have at most $m = \dim_{\mathbb{C}} X$ nonzero components. In other words, $\mathfrak{a}_r^k = \mathcal{A}^{0,k}(Y)$ for $k > m$.

To be more explicit, we write

$$\omega = \sum_{I \subset \{1, \dots, n+m\}} \omega_I d\bar{z}_I = \sum_{I \subset \{1, \dots, n\}} \omega_I d\bar{z}_I + \sum_{I \not\subset \{1, \dots, n\}} \omega_I d\bar{z}_I.$$

where $\omega_I \in C^\infty(Y)$. Note that the second term on the right hand side of the equality already lies in \mathfrak{a}_r^\bullet . Then

$$\mathcal{T}_r(\omega) = \sum_{I \subset \{1, \dots, n\}} \sum_{|\alpha| \leq r} \frac{1}{\alpha!} \partial^\alpha \omega_I|_X \cdot d\bar{z}_I \otimes z^\alpha.$$

A smooth function $f \in \mathcal{A}^{0,0}(Y)$ belongs to \mathfrak{a}_r^0 if and only if it is of the form

$$f = \sum_{|\alpha|=r+1} z^\alpha g_\alpha + \sum_{i=1}^n \bar{z}_i h_i$$

for some $g_\alpha, h_i \in C^\infty(Y)$. In general, $\omega \in \mathfrak{a}_r^\bullet$ if and only if

$$\omega \equiv \sum_{I \subset \{1, \dots, n\}} \omega_I d\bar{z}_I \pmod{d\bar{z}_{n+1}, \dots, d\bar{z}_m}$$

where $\omega_I \in \mathfrak{a}_r^0$. In other words, for $k \geq 1$,

$$\mathfrak{a}_r^k = \sum_{I \subset \{1, \dots, n\}} \mathfrak{a}_r^0 \cdot d\bar{z}_I + \sum_{i=n+1}^m d\bar{z}_i \wedge \mathcal{A}^{0,k-1}(Y).$$

Note that all the arguments above work if we substitute Y by any open subset $U \subset \mathbb{C}^{n+m}$ and X by the intersection of U with $0 \times \mathbb{C}^m$. So there are also isomorphisms on the level of sheaves:

$$\tilde{\mathcal{A}}_{X^{(r)}}^\bullet \cong \tilde{\mathcal{A}}_X^{0,\bullet} \otimes_{\mathbb{C}} \mathbb{C}[z_1, \dots, z_n] / (z_1, \dots, z_n)^{r+1}$$

and

$$\tilde{\mathcal{A}}_{X^{(\infty)}}^\bullet \cong \tilde{\mathcal{A}}_X^{0,\bullet}[[z_1, \dots, z_n]].$$

where $Y = U \subset \mathbb{C}^{n+m}$ and $X = U \cap (0 \times \mathbb{C}^m)$. Moreover, there is also an isomorphism

$$\mathcal{O}_{X_Y^{(r)}} \xrightarrow{\cong} \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[z_1, \dots, z_n] / (z_1, \dots, z_n)^{r+1}$$

defined in exactly the same way as that of $\tilde{\mathcal{T}}_r$, which fits into an isomorphism between two complexes of sheaves

$$0 \rightarrow \mathcal{O}_{X_Y^{(r)}} \rightarrow \tilde{\mathcal{A}}_{X_Y^{(r)}}^0 \xrightarrow{\bar{\partial}} \tilde{\mathcal{A}}_{X_Y^{(r)}}^1 \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \tilde{\mathcal{A}}_{X_Y^{(r)}}^m \rightarrow 0 \quad (2.6)$$

and

$$0 \rightarrow \mathcal{O}_X \otimes \mathcal{F}_r \rightarrow \tilde{\mathcal{A}}_X^{0,0} \otimes \mathcal{F}_r \xrightarrow{\bar{\partial} \otimes 1} \tilde{\mathcal{A}}_X^{0,1} \otimes \mathcal{F}_r \xrightarrow{\bar{\partial} \otimes 1} \dots \xrightarrow{\bar{\partial} \otimes 1} \tilde{\mathcal{A}}_X^{0,m} \otimes \mathcal{F}_r \rightarrow 0$$

where $\mathcal{F}_r = \mathbb{C}[z_1, \dots, z_n] / (z_1, \dots, z_n)^{r+1}$. The latter is exact by the Dolbeault-Grothendieck Lemma, thus we also have exactness of the first complex. Since exactness is a local property, we have the following proposition.

Proposition 2.4. *For any closed embedding $i : X \hookrightarrow Y$ of complex manifolds, the complex of sheaves $(\tilde{\mathcal{A}}_{X_Y^{(r)}}^\bullet, \bar{\partial})$ is a fine resolution of $\mathcal{O}_{X_Y^{(r)}}$ over X , where r is arbitrary nonnegative integer or ∞ .*

Proof. The case when r is finite was already taken care of by Example 2.3. For $r = \infty$, it suffices to show that, for any local chart U of Y and $V = U \cap X$ such that the pair (V, U) is biholomorphic to a pair of polydiscs (D', D) where $D \subset \mathbb{C}^{d+m}$ and $D' = D \cap 0 \times \mathbb{C}^m$, the complex of sections of (2.6) over V

$$0 \rightarrow \mathcal{O}_{X_Y^{(\infty)}}(\mathcal{V}) \rightarrow \tilde{\mathcal{A}}_{X_Y^{(\infty)}}^0(\mathcal{V}) \xrightarrow{\bar{\partial}} \tilde{\mathcal{A}}_{X_Y^{(\infty)}}^1(\mathcal{V}) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \tilde{\mathcal{A}}_{X_Y^{(\infty)}}^m(\mathcal{V}) \rightarrow 0$$

is exact, or equivalently, the sequence

$$0 \rightarrow \mathcal{O}_{D'_D^{(\infty)}}(D') \rightarrow \mathcal{A}^0(D'_D^{(\infty)}) \xrightarrow{\bar{\partial}} \mathcal{A}^1(D'_D^{(\infty)}) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^m(D'_D^{(\infty)}) \rightarrow 0 \quad (2.7)$$

is exact. Note that for any nonnegative integer s , the sequence

$$0 \rightarrow \mathcal{O}_{D'_D^{(s)}}(D') \rightarrow \mathcal{A}^0(D'_D^{(s)}) \xrightarrow{\bar{\partial}} \mathcal{A}^1(D'_D^{(s)}) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^m(D'_D^{(s)}) \rightarrow 0 \quad (2.8)$$

is isomorphic to

$$0 \rightarrow \mathcal{O}_{D'}(D') \otimes \mathcal{F}_r \rightarrow \mathcal{A}^{0,0}(D') \otimes \mathcal{F}_r \xrightarrow{\bar{\partial} \otimes 1} \cdots \xrightarrow{\bar{\partial} \otimes 1} \mathcal{A}^{0,m}(D') \otimes \mathcal{F}_r \rightarrow 0$$

by Example 2.3. The latter is exact by a stronger version of the Dolbeault-Grothendieck Lemma, thus so is (2.8). Let s vary and we obtain a inverse system of exact sequences. Since all the connecting homomorphisms are surjective, the inverse system satisfies the Mittag-Leffler condition and hence its inverse limit (2.7) is exact. \square

Corollary 2.5. *For any nonnegative integer r , the sheaf of dg-ideals $(\tilde{\mathfrak{a}}_r^\bullet, \bar{\partial})$ over Y is a fine resolution of \mathcal{I}^r .*

Proof. Think of $(\tilde{\mathcal{A}}_{X_Y^{(r)}}^\bullet, \bar{\partial})$ as a sheaf over Y , then we have a short exact sequence

$$0 \rightarrow (\tilde{\mathfrak{a}}_r^\bullet, \bar{\partial}) \rightarrow (\tilde{\mathcal{A}}_Y^{0,\bullet}, \bar{\partial}) \rightarrow (\tilde{\mathcal{A}}_{X_Y^{(r)}}^\bullet, \bar{\partial}) \rightarrow 0.$$

Together with the short exact sequence

$$0 \rightarrow \mathcal{I}^r \rightarrow \mathcal{O}_Y \rightarrow i_* \mathcal{O}_{X_Y^{(r)}} \rightarrow 0$$

we have a complex of short exact sequences of sheaves

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{I}^r & \longrightarrow & \mathcal{O}_Y & \longrightarrow & i_* \mathcal{O}_{X_Y^{(r)}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{\mathfrak{a}}_r^0 & \longrightarrow & \tilde{\mathcal{A}}_Y^{0,0} & \longrightarrow & \tilde{\mathcal{A}}_{X_Y^{(r)}}^0 \longrightarrow 0 \\
& & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} \\
0 & \longrightarrow & \tilde{\mathfrak{a}}_r^1 & \longrightarrow & \tilde{\mathcal{A}}_Y^{0,1} & \longrightarrow & \tilde{\mathcal{A}}_{X_Y^{(r)}}^1 \longrightarrow 0 \\
& & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} \\
\vdots & & \vdots & & \vdots & & \vdots \\
& & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} \\
0 & \longrightarrow & \tilde{\mathfrak{a}}_r^{\dim Y} & \longrightarrow & \tilde{\mathcal{A}}_Y^{0,\dim Y} & \longrightarrow & \tilde{\mathcal{A}}_{X_Y^{(r)}}^{\dim Y} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

The column in the middle is exact by Dolbeault-Grothendieck Lemma and so is the third column by Proposition 2.4. Consider the associated long exact sequence of cohomology sheaves of the diagram and we obtain exactness of the first column. \square

3. DIAGONAL EMBEDDING AND FORMAL GEOMETRY

Among all important and interesting examples is the diagonal embedding $\Delta : X \hookrightarrow X \times X$ of a complex manifold X into product of two copies of itself. We then have the formal neighborhood of the diagonal, $X_{X \times X}^{(\infty)}$, which is understood as the dga $(\mathcal{A}^{0,\bullet}(X_{X \times X}^{(\infty)}), \bar{\partial})$ constructed in the previous section. It is isomorphic to the Dolbeault resolution of the infinite holomorphic jet bundle \mathcal{J}_X^∞ . In this case one can write down the $\bar{\partial}$ -derivation explicitly (at least when X is Kähler) due to a theorem below by Kapranov ([?]), of which

we will reproduce the Kähler case in a slightly different way and discuss the general situation later.

Intuitively one would expect that there is an isomorphism

$$\mathcal{A}^{0,\bullet}(X_{X \times X}^{(\infty)}) \cong \mathcal{A}^{0,\bullet}(\hat{S}^\bullet(T^*X)) \quad (3.1)$$

by taking ‘Taylor expansions’, where $\hat{S}^\bullet(T^*X)$ is the bundle of complete symmetric tensor algebra generated by the cotangent bundle of X (which is natural identified with the conormal bundle of the diagonal embedding). Such an isomorphism does exist, but there is no canonical way to define it since one need to first choose some local coordinates to get Taylor expansions. Indeed we will see that the isomorphism depends (in a more or less 1-1 manner) on a smooth choice of formal (holomorphic) coordinates on X .

3.1. Differential geometry on formal discs. Fix a complex vector space V of dimension n . Consider the formal power series algebra

$$\mathcal{F} = \mathbb{C}[[V^*]] = \hat{S}(V^*) = \prod_{i \geq 0} S^i V^*,$$

that is, the function algebra of the formal neighborhood of $0 \in V$. It is a complete regular local algebra with a unique maximal ideal \mathfrak{m} . The associated graded algebra with respect to the \mathfrak{m} -filtration is the (uncompleted) symmetric algebra

$$\text{gr } \mathcal{F} = S(V^*) = \bigoplus_{i \geq 0} S^i V^*.$$

Since we are in the complex analytic situation, we endow \mathcal{F} with the canonical Fréchet topology. In algebraic setting, one need to use the \mathfrak{m} -adic topology on \mathcal{F} . However, the associated groups and spaces in question remain the same, though the topologies on them will be different. Since our arguments work for both Fréchet and \mathfrak{m} -adic settings, the topology will not be mentioned explicitly unless necessary. We also use $\hat{V} = \text{Spf } \mathcal{F}$ to denote the formal polydisc, either as a formal analytic space or a formal scheme.

Following the notations in [Ka99], we denote by $G^{(\infty)} = G^{(\infty)}(V)$ the proalgebraic group of automorphisms of the formal space \hat{V} , and by $J^{(\infty)} =$

$J^{(\infty)}(V)$ the normal subgroup consisting of those $\phi \in G^{(\infty)}$ with tangent map $d_0\phi = \text{Id}$ at 0. In other words, $J^{(\infty)}(V)$ is the kernel of $d_0 : G^{(\infty)} \rightarrow \mathbf{GL}_n(V)$. Let $\mathfrak{g}^{(\infty)} = \mathfrak{g}^{(\infty)}(V)$ and $\mathfrak{j}^{(\infty)} = \mathfrak{j}^{(\infty)}(V)$ be the corresponding Lie algebras. $\mathfrak{g}^{(\infty)}$ can be also interpreted as the Lie algebra of formal vector fields vanishing at 0, while $\mathfrak{j}^{(\infty)}$ is the Lie subalgebra of formal vector fields with vanishing constant and linear terms. We have decompositions

$$\mathfrak{g}^{(\infty)} = \prod_{i \geq 1} V \otimes S^i V^* = \prod_{i \geq 1} \text{Hom}(V^*, S^i V^*)$$

and

$$\mathfrak{j}^{(\infty)} = \prod_{i \geq 2} V \otimes S^i V^* = \prod_{i \geq 2} \text{Hom}(V^*, S^i V^*).$$

Elements of $\mathfrak{g}^{(\infty)}$ and $\mathfrak{j}^{(\infty)}$ acts on $\mathcal{F} = \prod_{i \geq 0} S^i V^*$ as derivations in the obvious way.

There is an exact sequence of proalgebraic groups

$$1 \rightarrow J^{(\infty)} \rightarrow G^{(\infty)} \rightarrow \mathbf{GL}_n(V) \rightarrow 1 \quad (3.2)$$

which canonically splits by regarding elements of $\mathbf{GL}_n = \mathbf{GL}_n(V)$ as jets of linear transformations on \widehat{V} . In other words, $G^{(\infty)}$ is a semidirect product:

$$G^{(\infty)} = J^{(\infty)} \rtimes \mathbf{GL}_n.$$

So we have a canonical bijection between sets

$$q : J^{(\infty)} \xrightarrow{\cong} G^{(\infty)} / \mathbf{GL}_n. \quad (3.3)$$

Moreover, there is a natural left $G^{(\infty)}$ -action on $J^{(\infty)}$ given by

$$(\phi, T) \cdot \varphi = \phi \circ T \circ \varphi \circ T^{-1}, \quad \forall (\phi, T) \in J^{(\infty)} \rtimes \mathbf{GL}_n = G^{(\infty)}, \forall \varphi \in J^{(\infty)}, \quad (3.4)$$

or equivalently,

$$\psi \cdot \varphi = \psi \circ \varphi \circ (d_0\psi)^{-1}, \quad \forall \psi \in G^{(\infty)}, \forall \varphi \in J^{(\infty)}, \quad (3.5)$$

which makes q into a $G^{(\infty)}$ -equivariant map. Moreover, an automorphism φ in $J^{(\infty)}$ can also be interpreted as a bijective morphism $\varphi : \widehat{T_0 V} \rightarrow \widehat{V}$, where

$\widehat{T_0V}$ is the completion of the tangent space T_0V at the origin, which is naturally identified with \widehat{V} itself. In addition, φ should induce the identity map on tangent spaces of two formal spaces at the origins. We call such a map φ as a *formal exponential map* since it satisfies analogous properties of exponential maps in classical Riemannian geometry. The above $G^{(\infty)}$ -action on $J^{(\infty)}$ then has a clearer meaning: think of $\psi \in G^{(\infty)}$ as a change of formal coordinates on \widehat{V} and the image of φ under this transformation is φ composed with $\psi : \widehat{V} \rightarrow \widehat{V}$ and precomposed with the inverse of the linearization of ψ on $\widehat{T_0V}$. In other words, $J^{(\infty)}$ is the set of all formal exponential maps $\varphi : \widehat{T_0V} \rightarrow \widehat{V}$, on which the action of $G^{(\infty)}$ comes from those on \widehat{V} and $\widehat{T_0V}$ with the latter being induced from the 'linearization map' $G^{(\infty)} \rightarrow \mathbf{GL}_n$.

To avoid confusion, we denote by

$$\mathcal{F}_T := \widehat{S}((T_0V)^*) = \widehat{S}(V^*)$$

the algebra of functions on $\widehat{T_0V}$ and by \mathfrak{m}_T its maximal ideal, though it is just another copy of \mathcal{F} . Having a formal exponential map $\varphi : \widehat{T_0V} \rightarrow \widehat{V}$ is the same as having an isomorphism of algebras

$$\varphi^* : \mathcal{F} \rightarrow \mathcal{F}_T \tag{3.6}$$

whose induced isomorphism between the associated graded algebras, which are both equal to $S(V^*)$, is the identity. As before, we want to set the $G^{(\infty)}$ -action on \mathcal{F}_T as the one induced from the linear action of \mathbf{GL}_n on \mathcal{F}_T and the projection $G^{(\infty)} \rightarrow \mathbf{GL}_n$.

Now let us give a another interpretation of $J^{(\infty)}$ as a $G^{(\infty)}$ -space. Consider the space $\mathcal{C}\text{onn}$ of flat torsion-free connections on \widehat{V} , that is, any element of $\mathcal{C}\text{onn}$ is a (nonlinear) map

$$\nabla : T\widehat{V} \rightarrow T\widehat{V} \otimes_{\mathcal{O}_{\widehat{V}}} T^*\widehat{V}$$

satisfying the Leibniz rule and equations defining flatness and torsion-freeness as in usual differential geometry, which we omit here. Most of time we also write ∇ for the induced connection on the cotangent bundle and its tensors

bundles:

$$\nabla : T^*\hat{V} \rightarrow T^*\hat{V} \otimes_{\mathcal{O}_{\hat{V}}} T^*\hat{V}.$$

For any complex vector space \hat{V} , there is a standard Euclidean connection $\nabla_{\mathfrak{E}}$ on \hat{V} , which acts as a left shift operator:

$$\nabla_{\mathfrak{E}} : \prod_{i \geq 0} V^* \otimes S^i V^* \rightarrow \prod_{i \geq 0} (V^*)^{\otimes 2} \otimes S^i V^*$$

where we identify $T^*\hat{V}$ with $V^* \otimes \hat{S}(V^*)$. Using $\nabla_{\mathfrak{E}}$ we can write down the components of $f \in \mathcal{F} = \prod_{i \geq 0} S^i V$ as

$$f = (\nabla_{\mathfrak{E}}^i f|_0)_{i \geq 0} = (f(0), \nabla_{\mathfrak{E}} f|_0, \nabla_{\mathfrak{E}}^2 f|_0, \dots) \in \mathcal{F} \quad (3.7)$$

where $\nabla_{\mathfrak{E}} f = df \in T^*\hat{V}$ is the usual differential of functions and by $\nabla_{\mathfrak{E}}^i f$ ($i \geq 2$) we mean $\nabla_{\mathfrak{E}}^{i-1} df$.

As in classical differential geometry, for each connection $\nabla \in \mathcal{C}^{\text{onn}}$ on the formal space \hat{V} we can assign an formal exponential map $\exp_{\nabla} : \widehat{T_0\hat{V}} \rightarrow \hat{V}$, which is defined to be the unique map in $J^{(\infty)}$ which pulls back ∇ into the Euclidean connection $\nabla_{\mathfrak{E}}$ on the completed tangent space, i.e.,

$$\exp_{\nabla}^* \nabla = \nabla_{\mathfrak{E}}. \quad (3.8)$$

To be more explicit, recall that giving a formal exponential map is the same as giving the way it pullbacks functions and so we define for each $f \in \mathcal{F}$,

$$\exp_{\nabla}^*(f) = (\nabla^i f|_0)_{i \geq 0} = (f(0), \nabla f|_0, \nabla^2 f|_0, \dots) \in \prod_{i \geq 0} S^i(T_0V)^* = \mathcal{F}_T \quad (3.9)$$

which is analogous to (3.7). Here again $\nabla f = df$ and $\nabla^i f = \nabla^{i-1} df$ for $i \geq 2$. Note that to make sure the terms in the expression lie in symmetric tensors one has to resort to the torsionfreeness and flatness of ∇ . It is an exercise to check by (3.7) that such defined \exp_{∇} satisfies the property (3.8) and conversely, any formal exponential map satisfying (3.8) must be of the form (3.9).

On the other hand, any $\phi \in J^{(\infty)}$ pushforwards the Euclidean connection $\nabla_{\mathfrak{E}}$ on $\widehat{T_0\hat{V}}$ into a flat torsion-free connection $\nabla = \phi_* \nabla_{\mathfrak{E}}$. Thus by the

discussion above we obtain a bijection

$$\exp : \mathcal{C}\text{onn} \xrightarrow{\cong} J^{(\infty)}. \quad (3.10)$$

Note that $G^{(\infty)}$ naturally acts on $\mathcal{C}\text{onn}$ from left by pushforwarding connections via automorphisms of \widehat{V} . Recall that we also have a $G^{(\infty)}$ action on $J^{(\infty)}$ defined by (3.5).

Lemma 3.1. *The map $\exp : \mathcal{C}\text{onn} \rightarrow J^{(\infty)}$ defined above is $G^{(\infty)}$ -equivariant.*

Proof. One only needs to notice the following commutative diagram of spaces equipped with connections

$$\begin{array}{ccc} (\widehat{T_0 V}, \partial) & \xrightarrow{d_0 \varphi} & (\widehat{T_0 V}, \partial) \\ \downarrow \exp_{\nabla} & & \downarrow \exp_{\varphi_* \nabla} \\ (\widehat{V}, \nabla) & \xrightarrow{\varphi} & (\widehat{V}, \varphi_* \nabla) \end{array}$$

which corresponds exactly to (3.5). \square

All the arguments remain valid for $V^{(r)} = \text{Spf } \mathcal{F}/\mathfrak{m}^r$, the r -th order formal neighborhood of V , where \mathfrak{m} is the maximal ideal of the local algebra \mathcal{F} . One can also define $G^{(r)}$, $J^{(r)}$, etc., and state similar results about them. We leave the details to the reader.

3.2. Formal geometry. We now introduce the bundle of formal coordinate systems $p : X_{\text{coord}} \rightarrow X$ of a smooth complex manifold X . By definition (§4.4., [Ka99]), for $x \in X$ the fiber $X_{\text{coord},x}$ is the space of infinite jets of biholomorphisms $\varphi : V \cong \mathbb{C}^n \rightarrow X$ with $\varphi(0) = x$. Hence X_{coord} is a natural holomorphic principal $G^{(\infty)}$ -bundle.

We have a canonical isomorphism between sheaves of algebras

$$X_{\text{coord}} \times_X \mathcal{J}_X^\infty \cong X_{\text{coord}} \times \mathcal{F}. \quad (3.11)$$

over X_{coord} . Thus X_{coord} can be characterized by the following universal property ([BeKa04], [?]): given any complex space S , a morphism $\eta : S \rightarrow X$ and an isomorphism $\zeta : \eta^* \mathcal{J}_X^\infty \cong \mathcal{O}_S \widehat{\otimes} \mathcal{F}$ of sheaves of topological algebras over S , there is a unique morphism $\eta' : S \rightarrow X_{\text{coord}}$ such that $\eta = p \circ \eta'$

and ζ is induced from the isomorphism (3.11). Note that such ζ does not necessarily exist for arbitrary $\eta : S \rightarrow X$, but always does for those S which are Stein.

One can obtain various canonical jet bundles on X by applying Borel constructions on principal $G^{(\infty)}$ -bundle X_{coord} . For example, \mathcal{F} is naturally a $G^{(\infty)}$ -module and the corresponding sheaf of algebras associated to the $G^{(\infty)}$ -torsor X_{coord} coincides with the jet bundle of holomorphic functions \mathcal{J}_X^∞ :

$$X_{\text{coord}} \times_{G^{(\infty)}} \mathcal{F} \cong \mathcal{J}_X^\infty$$

To get the natural flat connection on \mathcal{J}_X^∞ , one could adopt the language of Harish-Chandra torsors as in [BeKa04]. We omit it since this connection will not be used in this paper. Other jet bundles, such as $\mathcal{J}^\infty T_X$, the jet bundle of the tangent bundle, and $\mathcal{J}^\infty T_X^*$, the jet bundle of cotangent bundle, can be obtained in a similar way. Again we refer interested readers to [BeKa04].

Another related space which is at the very core of our discussions is the bundle of formal exponential maps introduced in [Ka99], which we denote by X_{exp} . Each fiber $X_{\text{exp},x}$ at $x \in X$ is the space of jets of holomorphic maps $\phi : T_x X \rightarrow X$ such that $\phi(0) = x$, $d_0 \phi = \text{Id}$. We have a map

$$X_{\text{coord}} \rightarrow X_{\text{exp}}, \quad \phi \mapsto \phi \circ (d_0 \phi)^{-1}$$

which induce a biholomorphism

$$X_{\text{coord}}/\mathbf{GL}_n \cong X_{\text{exp}} \tag{3.12}$$

On the other hand, we can define the bundle of jets of flat torsion-free connection

$$X_{\text{conn}} = X_{\text{coord}} \times_{G^{(\infty)}} \mathfrak{Conn}$$

whose fiber at a given point $x \in X$ consists of all flat torsion-free connections on the formal neighborhood of x . By combining the $G^{(\infty)}$ -equivariant bijections (3.3), (3.10) and (3.12), we get

$$X_{\text{conn}} \cong X_{\text{coord}} \times_{G^{(\infty)}} \mathcal{J}^{(\infty)} \cong X_{\text{coord}} \times_{G^{(\infty)}} G^{(\infty)}/\mathbf{GL}_n \cong X_{\text{exp}}. \tag{3.13}$$

In other words, we can naturally identify jet bundle of flat torsion-free connections with the jet bundle of formal exponential maps. From now on, we will not distinguish between these two jet bundles and denote both by X_{conn} , though both interpretations will be adopted in the rest of the paper.

By definition of X_{conn} , there is a tautological flat and torsion-free connection over X_{conn} ,

$$\nabla_{\text{tau}} : \pi^* \mathcal{J}^\infty T^*X \rightarrow \pi^* \mathcal{J}^\infty T^*X \otimes_{\pi^* \mathcal{J}_X^\infty} \pi^* \mathcal{J}^\infty T^*X,$$

which is $\mathcal{O}_{X_{\text{coord}}}$ -linear yet satisfies the Leibniz rule with respect to the differential

$$\tilde{d}^{(\infty)} : \pi^* \mathcal{J}_X^\infty \rightarrow \pi^* \mathcal{J}^\infty T^*X$$

that is the pullback of

$$d^{(\infty)} : \mathcal{J}_X^\infty \rightarrow \mathcal{J}^\infty T^*X.$$

Here $d^{(\infty)}$ is a \mathcal{O}_X -linear differential obtained by apply the Borel construction with X_{coord} and the differential $d : \mathcal{O}_{\hat{V}} \rightarrow T^*\hat{V}$ on the formal disc.

On the other hand, since X_{conn} can also be interpreted as the bundle of formal exponential maps, we have a tautological isomorphism between sheaves of algebras over X_{conn} ,

$$\text{Exp}^* : \pi^*(X_{\text{coord}} \times_{G^{(\infty)}} \mathcal{F}) \rightarrow \pi^*(X_{\text{coord}} \times_{G^{(\infty)}} \mathcal{F}_T).$$

The domain is identified as $\pi^* \mathcal{J}_X^\infty$ or $\mathcal{O}_{X_{X \times X}^{(\infty)}}$, the structure sheaf of the formal neighborhood of the diagonal in $X \times X$, while for the codomain we have

$$X_{\text{coord}} \times_{G^{(\infty)}} \mathcal{F}_T \cong X_{\text{coord}} \times_{G^{(\infty)}} \mathbf{GL}_n \times_{\mathbf{GL}_n} \mathcal{F}_T \cong X_{\text{coord}}/J^{(\infty)} \times_{\mathbf{GL}_n} \mathcal{F}_T$$

by our definition of the $G^{(\infty)}$ -action on \mathcal{F}_T . But the principal \mathbf{GL}_n -bundle $X_{\text{coord}}/J^{(\infty)}$ is exactly the bundle of (0th-order) frames on X , so

$$X_{\text{coord}}/J^{(\infty)} \times_{\mathbf{GL}_n} V \cong TX.$$

Since the \mathbf{GL}_n action respects the decomposition $\mathcal{F}_T = \prod_{i \geq 0} S^i V^*$, we get

$$X_{\text{coord}}/J^{(\infty)} \times_{\mathbf{GL}_n} \mathcal{F}_T \cong \prod_{i \geq 0} S^i T^*X = \hat{S}(T^*X),$$

which is the structure sheaf of $X_{TX}^{(\infty)}$, the formal neighborhood of the zero section of TX. In short, we have a *tautological exponential map*

$$\text{Exp} : \pi^* X_{TX}^{(\infty)} \rightarrow \pi^* X_{X \times X}^{(\infty)}$$

or equivalently, an isomorphism of bundles of topological algebras

$$\text{Exp}^* : \pi^* \mathcal{O}_{X_{X \times X}^{(\infty)}} \rightarrow \pi^* \mathcal{O}_{X_{TX}^{(\infty)}}.$$

which we might call as the *tautological Taylor expansion map*. Moreover, the map induced by Exp^* between associated bundle of graded algebras

$$\text{gr Exp}^* : \pi^* \text{gr } \mathcal{O}_{X_{X \times X}^{(\infty)}} = \pi^* S(T^*X) \rightarrow \pi^* S(T^*X)$$

is the identity map. In virtue of (3.9), Exp^* can be written in terms of ∇_{tau} :

$$\text{Exp}^*(f) = (\nabla_{\text{tau}}^i f|_0)_{i \geq 0} = (f(0), \nabla_{\text{tau}} f|_0, \nabla_{\text{tau}}^2 f|_0, \dots) \in \pi^* \hat{S}(T^*X) \quad (3.14)$$

where the 'restriction to the origin' map $\pi^* S^i \mathcal{J}^\infty T^*X \rightarrow \pi^* S^i T^*X$ comes from the local restriction map $T^*\hat{V} \rightarrow T_0^*\hat{V} = V^*$ by applying Borel construction with X_{coord} and then pulling back onto X_{conn} via π . Again $\nabla_{\text{tau}} f$ means $d^{(\infty)} f$ and so on.

Remark 3.2. Note that there is no natural $G^{(\infty)}$ - or $J^{(\infty)}$ -action on X_{conn} . Yet it is a torsor over the proalgebraic group bundle $J^{(\infty)}(TX)$, whose fiber over $x \in X$ is the group of jets of biholomorphisms $\varphi : T_x X \rightarrow T_x X$ with $\varphi(0) = 0$, $d_0 \varphi = \text{Id}$. Indeed, consider the proalgebraic group $\text{Aut}_0 \widehat{T_0 V}$ of automorphisms of $\widehat{T_0 V}$ whose tangent maps are identity. $\text{Aut}_0 \widehat{T_0 V}$ can be identified $J^{(\infty)}$ as sets, yet we endow it with a different $G^{(\infty)}$ -action which is the conjugation of the one on $\widehat{T_0 V}$ induced from $G^{(\infty)} \rightarrow \mathbf{GL}_n$. $\text{Aut}_0 \widehat{T_0 V}$ acts on $J^{(\infty)}$ from right by precomposition and the action is compatible with the $G^{(\infty)}$ -actions. Finally, notice that

$$J^{(\infty)}(TX) \cong X_{\text{coord}} \times_{G^{(\infty)}} \text{Aut}_0 \widehat{T_0 V}.$$

Let $\mathfrak{j}^{(\infty)}(\mathrm{TX})$ be the bundle of Lie algebras associated to $\mathrm{J}^{(\infty)}(\mathrm{TX})$. We have a natural splitting

$$\mathfrak{j}^{(\infty)}(\mathrm{TX}) = \prod_{i \geq 2} \mathrm{Hom}(S^i \mathrm{TX}, \mathrm{TX}) = \prod_{i \geq 2} \mathrm{Hom}(T^* \mathrm{X}, S^i T^* \mathrm{X}). \quad (3.15)$$

Remark 3.3. All the jet bundles we are discussing here are holomorphic and although in general they might not admit global holomorphic sections, there always exist global smooth sections. For example, let us consider the fiber bundle $\pi_n : X_{\mathrm{conn}}^{(n)}(\mathrm{X}) \rightarrow \mathrm{X}$ of 'n-th order exponential maps', cf. §4.2., [Ka99]. By definition, for $x \in \mathrm{X}$ the fiber $X_{\mathrm{conn},x}^{(n)}$ is the space of n-th order jets of holomorphic maps $\phi : T_x \mathrm{X} \rightarrow \mathrm{X}$ such that $\phi(0) = x$, $d_0 \phi = \mathrm{Id}$. Thus we have a chain of projections

$$\mathrm{X} \leftarrow X_{\mathrm{conn}}^{(2)} \leftarrow X_{\mathrm{conn}}^{(3)} \leftarrow \cdots. \quad (3.16)$$

Each $X_{\mathrm{conn}}^{(n+1)}$ is an affine bundle over $X_{\mathrm{conn}}^{(n)}$ whose associated vector bundle is $\pi_n^* \mathrm{Hom}(S^{n+1} \mathrm{TX}, \mathrm{TX})$, so any section of $X_{\mathrm{conn}}^{(n)}$ can be lifted to a smooth section of the next bundle in the diagram. The inverse limit of the diagram is exactly the bundle $\pi : X_{\mathrm{conn}} \rightarrow \mathrm{X}$. Thus X_{conn} admits global smooth sections.

Given any smooth section σ of X_{conn} , together with the Taylor expansion map Exp^* , it induces a smooth homomorphism

$$\mathrm{exp}_\sigma^* : \mathcal{J}_X^\infty \rightarrow \widehat{S}(T^* \mathrm{X})$$

of bundles of algebras over X . It is holomorphic if and only if σ is holomorphic. Conversely, X_{conn} satisfies universal properties similar to the one of X_{coord} . Namely, given a holomorphic (resp. smooth) map $\eta : S \rightarrow \mathrm{X}$, any holomorphic (resp. smooth) isomorphism $\zeta : \eta^* \mathcal{J}_X^\infty \rightarrow \eta^* \widehat{S}(T^* \mathrm{X})$, which induces the identity map on the associated graded algebras

$$\mathrm{gr} \mathcal{J}_X^\infty = S(T^* \mathrm{X}) = \mathrm{gr} \widehat{S}(T^* \mathrm{X}),$$

arises in the same way from a unique holomorphic (resp. smooth) section $\eta' : S \rightarrow X_{\mathrm{conn}}$ such that $\eta = \pi \circ \eta'$.

In particular, global smooth sections of X_{conn} correspond in a 1 – 1 manner to all possible smooth isomorphisms between \mathcal{J}_X^∞ and $\hat{S}(T^*X)$ which induce identity map on associated graded algebras. As a (nonlinear) holomorphic bundle, X_{conn} carries a flat $(0, 1)$ -connection \bar{d} , such that for any given smooth section σ of X_{conn} , its anti-holomorphic differential

$$\omega_\sigma := \bar{d}\sigma \in \mathcal{A}^{0,1}(\mathfrak{j}^{(\infty)}(TX)) \quad (3.17)$$

is well-defined and it satisfies a Maurer-Cartan type equation

$$\bar{d}\omega_\sigma + \frac{1}{2}[\omega_\sigma, \omega_\sigma] = 0. \quad (3.18)$$

Projecting ω to the n -th graded component in (3.15), we get $(0, 1)$ -forms

$$\alpha_\sigma^n \in \mathcal{A}_X^{0,1}(\text{Hom}(S^n TX, TX)) = \mathcal{A}_X^{0,1}(\text{Hom}(T^*X, S^n T^*X)).$$

Moreover, one can extend the Taylor expansion map \exp_σ^* linearly with respect to $\mathcal{A}^{0,\bullet}(X)$ to a homomorphism

$$\exp_\sigma^* : \mathcal{A}_X^{0,\bullet}(\mathcal{J}_X^\infty) \rightarrow \mathcal{A}_X^{0,\bullet}(\hat{S}(T^*X)) \quad (3.19)$$

or

$$\exp_\sigma^* : \mathcal{A}^\bullet(X_{X \times X}^{(\infty)}) \rightarrow \mathcal{A}_X^{0,\bullet}(\hat{S}(T^*X)) \quad (3.20)$$

between graded algebras, yet it does not commute with the \bar{d} -differentials. The deficiency is measured exactly by ω , that is,

$$\omega_\sigma = \bar{d} \exp_\sigma^* \circ (\exp_\sigma^*)^{-1}, \quad (3.21)$$

where $\bar{d} \exp_\sigma^* = [\bar{d}, \exp_\sigma^*]$. This suggests that we can correct the usual holomorphic structure on $\hat{S}(T^*X)$ to make \exp_σ^* holomorphic. Let $\tilde{\omega}$ and $\tilde{\alpha}_\sigma^n$ be the odd derivations of the graded algebra $\mathcal{A}^{0,\bullet}(\hat{S}(T^*X))$ induced by ω and α_σ^n respectively. Define a new differential $D_\sigma = \bar{d} - \tilde{\omega} = \bar{d} - \sum_{n \geq 2} \tilde{\alpha}_\sigma^n$, then we have the following result:

Proposition 3.4. *The Taylor expansion map with respect to any given smooth section σ of X_{conn}*

$$\exp_\sigma^* : (\mathcal{A}^\bullet(X_{X \times X}^{(\infty)}), \bar{d}) \rightarrow (\mathcal{A}^{0,\bullet}(\hat{S}(T^*X)), D_\sigma)$$

is an isomorphism of *dgas*. In particular $D_\sigma^2 = 0$, which is equivalent to the Maurer-Cartan equation (3.18).

3.3. The Kähler case: Kapranov's result revisited. Now suppose that X is equipped with a Kähler metric h . Let ∇ be the canonical $(1, 0)$ -connection in TX associated with h , so that

$$[\nabla, \nabla] = 0 \text{ in } \mathcal{A}_X^{2,0}(\text{End}(TX)). \quad (3.22)$$

and it is torsion-free, which is equivalent to the condition for h to be Kähler.

Set $\tilde{\nabla} = \nabla + \bar{\partial}$, where $\bar{\partial}$ is the $(0, 1)$ -connection defining the complex structure. The curvature of $\tilde{\nabla}$ is just

$$R = [\bar{\partial}, \nabla] \in \mathcal{A}_X^{1,1}(\text{End}(TX)) = \mathcal{A}_X^{0,1}(\text{Hom}(TX \otimes TX, TX)) \quad (3.23)$$

which is a Dolbeault representative of the Atiyah class α_{TX} of the tangent bundle. In particular one has the Bianchi identity:

$$\bar{\partial}R = 0 \text{ in } \mathcal{A}_X^{0,2}(\text{Hom}(TX \otimes TX, TX))$$

Actually, by the torsion-freeness we have

$$R \in \mathcal{A}_X^{0,1}(\text{Hom}(S^2TX, TX))$$

Now define tensor fields R_n , $n \geq 2$, as higher covariant derivatives of the curvature:

$$R_n \in \mathcal{A}_X^{0,1}(\text{Hom}(S^2TX \otimes TX^{\otimes(n-2)}, TX)), \quad R_2 := R, \quad R_{i+1} = \nabla R_i \quad (3.24)$$

In fact R_n is totally symmetric, i.e.,

$$R_n \in \mathcal{A}_X^{0,1}(\text{Hom}(S^nTX, TX)) = \mathcal{A}_X^{0,1}(\text{Hom}(T^*X, S^nT^*X))$$

by the flatness of ∇ (3.22). Note that if we think of ∇ as the induced connection on the cotangent bundle, the same formulas (3.23) and (3.24) give $-R_n$.

The connection ∇ determines a smooth section of X_{conn} , which we write as $\sigma = [\nabla]_\infty$, by assigning to each point $x \in X$ the holomorphic jets of ∇ .

This has been done implicitly in the proof of Lemma 2.9.1., [Ka99]. One can check that the induced Taylor expansion map

$$\exp_\sigma^* : \mathcal{A}^\bullet(X_{X \times X}^{(\infty)}) \xrightarrow{\cong} \mathcal{A}_X^{0,\bullet}(\hat{S}(T^*X))$$

by

$$\exp_\sigma^*([\eta]_\infty) = (\Delta^*\eta, \Delta^*\nabla\eta, \Delta^*\nabla^2\eta, \dots, \Delta^*\nabla^n\eta, \dots) \in \mathcal{A}_X^{0,\bullet}(\hat{S}(T^*X)) \quad (3.25)$$

for any $[\eta]_\infty \in \mathcal{A}^\bullet(X_{X \times X}^{(\infty)})$. Here ∇ is understood as the pullback of ∇ (on the cotangent bundle) via pr_2 , which is a constant family of connections along fibers of pr_1 , instead of jets of ∇ . The key observation is that the right hand side of the formula only depends on the class $[\eta]_\infty \in \mathcal{A}^\bullet(X_{X \times X}^{(\infty)})$ and the holomorphic jets of the ∇ .

Theorem 3.5 (Theorem 2.8.2, [Ka99]). *Assume X is Kähler. With the notations from §?? and above, we have $\alpha_\sigma^n = -R_n$, i.e., $\omega_\sigma = -\sum_{n \geq 2} R_n$. Thus there is an isomorphism between dgas*

$$\exp_\sigma^* : (\mathcal{A}^\bullet(X_{X \times X}^{(\infty)}), \bar{\partial}) \rightarrow (\mathcal{A}_X^{0,\bullet}(\hat{S}(T^*X)), D_\sigma)$$

The derivation $D_\sigma = \bar{\partial} + \sum_{n \geq 2} \tilde{R}_n$, where \tilde{R}_n is the odd derivation of $\mathcal{A}^{0,\bullet}(\hat{S}(T^*X))$ induced by R_n .

Proof. By the discussion at the end of § ?? and equality (3.21), one only needs to compute $\bar{\partial} \exp_\sigma^* \circ (\exp_\sigma^*)^{-1}$. Let

$$\Upsilon = [\bar{\partial}, \nabla] \in \mathcal{A}_{X \times X}^{0,1}(\text{Hom}(\text{pr}_2^*T^*X, S^n(\text{pr}_2^*T^*X)))$$

and

$$\tilde{\Upsilon} \in \mathcal{A}_{X \times X}^{0,1}(\text{Hom}(\hat{S}^\bullet(\text{pr}_2^*T^*X), \hat{S}^{\bullet+1}(\text{pr}_2^*T^*X)))$$

the derivation on $\mathcal{A}_{X \times X}^{0,\bullet}(\hat{S}^\bullet(\text{pr}_2^*T^*X))$ induced by Υ which increases the degree on \hat{S}^\bullet by 1. Here again by ∇ we mean the pullback $\text{pr}_2^*\nabla$. But it is constant in the direction of the first factor of $X \times X$. So if we denote by $\bar{\partial} = \bar{\partial}_1 + \bar{\partial}_2$ the natural splitting of $\bar{\partial}$ on $X \times X$ with respect of the product structure, we get

$$\Upsilon = [\bar{\partial}, \nabla] = [\bar{\partial}_2, \nabla] = \text{pr}_2^*(-R)$$

For any $[f]_\infty \in \mathcal{A}^0(X_{X \times X}^{(\infty)})$, we have

$$\nabla^n \bar{\partial} f = \bar{\partial} \nabla^n f - \sum_{i+j=n-2} \nabla^i \circ \tilde{\gamma} \circ \nabla^j (\nabla f)$$

By evaluating $\tilde{\gamma}$ and expanding the action of ∇^i via the Leibniz rule, we find

$$\nabla^n \bar{\partial} f = \bar{\partial} \nabla^n f - \sum_{i=1}^{n-1} \nabla^{n-i+1} \gamma \circ \nabla^i f$$

Finally apply both sides by Δ^* and we obtain

$$\alpha_\sigma^n = \Delta^* \nabla^{n-2} \gamma = \Delta^* \text{pr}_2^* (-\nabla^{n-2} \mathbf{R}) = -\mathbf{R}_n.$$

The last statement of the theorem follows immediately from Proposition 3.4. \square

Remark 3.6. Our notations are slightly different from that of Kapranov, who wrote $\mathbf{R}_n^* \in \mathcal{A}^{0,1}(\text{Hom}(T^*X, S^n T^*X))$ as the transpose of \mathbf{R}_n while in this paper the same symbol stands for both.

We conclude this section by a slightly more generalized version of Theorem 3.5, which will be used later. Suppose $f : X \rightarrow Y$ is an arbitrary holomorphic map. We consider the graph of f

$$\tilde{f} = (\text{Id}, f) : X \rightarrow X \times Y$$

which is an embedding. So we can consider the formal neighborhood $X_{X \times Y}^{(\infty)}$. All the constructions above can be carried out in exactly the same way with only slight adjustment and give us a description of the Dolbeault dga $\mathcal{A}^\bullet(X_{X \times Y}^{(\infty)})$. Namely, consider the pullback bundle f^*Y_{conn} over X . By the universal property of Y_{conn} , each smooth section σ of f^*Y_{conn} naturally corresponds to an isomorphism

$$\eta_\sigma : \mathcal{A}^\bullet(X_{X \times Y}^{(\infty)}) \rightarrow \mathcal{A}_X^{0,\bullet}(\hat{S}(f^*T^*Y))$$

of graded algebras. One can also view sections of f^*Y_{conn} as jets of flat torsion-free connections on $X_{X \times Y}^{(\infty)}$ along Y -fibers. In particular, when Y carries a Kähler metric and the canonical $(1,0)$ -connection ∇ , we can pullback

∇ via the projection $X \times Y \rightarrow Y$, which determines a smooth section σ of f^*Y_{conn} and a Taylor expansion map

$$\exp_\sigma^* : \mathcal{A}^\bullet(X_{X \times Y}^{(\infty)}) \rightarrow \mathcal{A}_X^{0,\bullet}(\hat{S}(f^*T^*Y)), \quad [\zeta]_\infty \mapsto (\tilde{f}^*\zeta, \tilde{f}^*\nabla\zeta, \tilde{f}^*\nabla^2\zeta, \dots),$$

for any $[\zeta]_\infty \in \mathcal{A}^\bullet(X_{X \times Y}^{(\infty)})$. By abuse of notations, we still write

$$R_n \in \mathcal{A}_Z^{0,1}(\text{Hom}(S^n(f^*TY), f^*TY))$$

as the pullback of the curvature form of Y and its covariant derivatives via f . Then we have the following theorem,

Theorem 3.7. *We have an isomorphism between dgas*

$$\exp_\sigma^* : (\mathcal{A}^\bullet(X_{X \times Y}^{(\infty)}), \bar{\partial}) \xrightarrow{\cong} (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(f^*T^*Y)), D_\sigma)$$

where $D_\sigma = \bar{\partial} + \sum_{n \geq 2} \tilde{R}_n$ and \tilde{R}_n is the derivation of degree $+1$ induced by R_n .

4. GENERAL CASE OF ARBITRARY EMBEDDINGS

Let $i : X \hookrightarrow Y$ be an arbitrary embedding and $\mathcal{A}^\bullet(X_Y^{(\infty)})$ the Dolbeault dga associated to the formal neighborhood of X inside Y as in § 3.2. The goal is to build some appropriate isomorphism $\mathcal{A}^\bullet(X_Y^{(\infty)}) \cong \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee))$ and write down the $\bar{\partial}$ -derivation explicitly under such identification. We will show that this can be derived from the special yet universal case considered in § 3.

4.1. Differential geometry of complex submanifolds.

4.1.1. *Splitting of normal exact sequence and Kodaira-Spencer class.* Over X we have the exact sequence of holomorphic vector bundles defining the normal bundle $N_{X/Y}$

$$0 \rightarrow TX \xrightarrow{\iota} i^*TY \xrightarrow{p} N_{X/Y} \rightarrow 0 \quad (4.1)$$

and its dual

$$0 \rightarrow N_{X/Y}^\vee \xrightarrow{p^\vee} i^*T^*Y \xrightarrow{\iota^\vee} T^*X \rightarrow 0 \quad (4.2)$$

We fix a choice of C^∞ -splitting of the normal exact sequence (4.1), i.e., two smooth homomorphisms of vector bundles $\tau : i^*TY \rightarrow TX$ and $\rho : N_{X/Y} \rightarrow i^*TY$ satisfying

$$\tau \circ \iota = \text{Id}_{TX}, \quad \rho \circ \rho = \text{Id}_{N_{X/Y}}, \quad \iota \circ \tau + \rho \circ \rho = \text{Id}_{i^*TY}$$

and denote the corresponding dual splitting on the conormal exact sequence (4.2) by $\tau^\vee : T^*X \rightarrow i^*T^*Y$ and $\rho^\vee : i^*T^*Y \rightarrow N_{X/Y}^\vee$. We can choose the splittings as the orthonormal decomposition induced by the Kähler metric on Y (if there is one), but again we will never need the metric explicitly in our discussion.

Think of τ^\vee as a C^∞ -section of the holomorphic vector bundle

$$\text{Hom}(T^*X, i^*T^*Y),$$

we can form

$$\beta_{X/Y} := \bar{\partial}\tau^\vee \in \mathcal{A}_X^{0,1}(\text{Hom}(T^*X, T^*Y)).$$

In fact

$$\beta_{X/Y} \in \mathcal{A}_X^{0,1}(\text{Hom}(T^*X, N_{X/Y}^\vee)).$$

To see this, just apply $\bar{\partial}$ on both sides of equality $\iota^\vee \circ \tau^\vee = \text{Id}_{T^*X}$ and note that ι^\vee is holomorphic. By definition

$$\bar{\partial}\beta_{X/Y} = 0 \quad \text{in} \quad \mathcal{A}^{0,2}(\text{Hom}(T^*X, N_{X/Y}^\vee)),$$

hence $\beta_{X/Y}$ defines a cohomology class $[\beta_{X/Y}] \in \text{Ext}_X^1(T^*X, N_{X/Y}^\vee)$, which is the obstruction class for the existence of a holomorphic splitting of the exact sequence (4.2) or (4.1). Let's call it Kodaira-Spencer class. Finally also note that

$$\beta_{X/Y} = -\bar{\partial}\rho = -\bar{\partial}\rho^\vee \in \mathcal{A}^{0,1}(\text{Hom}(N_{X/Y}, TX)) = \mathcal{A}^{0,1}(\text{Hom}(T^*X, N_{X/Y}^\vee)) \quad (4.3)$$

4.1.2. *Shape operator.* Suppose ∇ is an arbitrary $(1, 0)$ -connection on TY , without any additional assumption. We use the same notation for the pullback connection on i^*TY or i^*T^*Y over X . The induced connection on the normal bundle via the chosen splitting is denoted by ∇^\perp :

$$\nabla_V^\perp \mu := p(\nabla_V \rho(\mu)) \in C_X^\infty(N_{X/Y}), \quad \forall \mu \in C_X^\infty(N_{X/Y}), V \in C^\infty(TX).$$

(here we identify $T^{1,0}X$ with TX) Analogous to the shape operator in Riemannian geometry, we also define a linear operator $A : TX \otimes N_{X/Y} \rightarrow TX$ by

$$A^\sharp(V) = -\tau(\nabla_V \rho(\mu)), \quad \forall \mu \in C_X^\infty(N_{X/Y}), V \in C^\infty(TX). \quad (4.4)$$

That is, we first lift a smooth section μ of the normal bundle to a section of i^*TY via the splitting, then take its derivatives with respect to the induced connection on i^*TY and finally project the output onto TX . Note that A is not a holomorphic map between vector bundles.

4.2. Taylor expansions in normal direction.

4.2.1. *General discussions.* Let $i : X \hookrightarrow Y$ be an embedding where Y is not necessarily Kähler. Similar to what has been done in § 3.2, we can consider all isomorphisms

$$\mathcal{A}^\bullet(X_Y^{(\infty)}) \xrightarrow{\cong} \mathcal{A}_X^{0,\bullet}(\hat{S}(N_{X/Y}^\vee))$$

which induces identity on the associated graded bundle $S(N_{X/Y}^\vee)$ all at once and there is a infinite dimensional bundle $\Psi_{X/Y} \rightarrow X$ whose smooth sections correspond exactly to such isomorphisms. Indeed, for each $x \in X$, the fiber $\Psi_{X/Y,x}$ is the space of jets of holomorphic maps $\psi : N_{X/Y,x} \rightarrow Y$ with $\psi(0) = x$ and $p \circ d_0\psi = \text{Id}$, where $p : i^*TY \rightarrow N_{Y/X}$ is the natural quotient map.

Similarly, we can define another bundle $\Theta_{X/Y}$ over X whose fiber at x is the space of jets of maps $\theta : N_{X/Y,x} \rightarrow T_x Y$ with $\theta(0) = x$ and $p \circ d_0\theta = \text{Id}$. Then $\Theta_{X/Y}$ admits a natural action of $J^{(\infty)}(T^*Y)$ (or more precisely, $J^{(\infty)}(T^*Y|_X)$) from left. In fact, we have

$$\Psi_{X/Y} = Y_{\text{conn}}|_X \times_{J^{(\infty)}(T^*Y)} \Theta_{X/Y}.$$

For our purpose here, however, it is not quite convenient to deal with $\Psi_{X/Y}$ since even we have already understood Y_{conn} in various geometric ways, general sections of the bundle $\Theta_{X/Y}$ are difficult to handle. So instead we only look at linear liftings $N_{X/Y} \rightarrow TY$ which form a subbundle $\Theta_{X/Y}^{(1)} \subset \Theta_{X/Y}$. Moreover, there is a canonical retraction $\Theta_{X/Y} \rightarrow \Theta_{X/Y}^{(1)}$ sending jets of maps $\psi : N_{X/Y,x} \hookrightarrow Y$ to their linearizations $d_0\psi$. Thus we have a fiberwise surjection of bundles

$$\kappa : Y_{\text{conn}}|_X \times_X \Theta_{X/Y}^{(1)} \rightarrow Y_{\text{conn}}|_X \times_{J^{(\infty)}(T^*Y)} \Theta_{X/Y} = \Psi_{X/Y}$$

As an affine bundle over the vector bundle $N_{X/Y}^\vee \otimes TX$, $\Theta_{X/Y}^{(1)}$ admits smooth sections which are just C^∞ -liftings $\rho : N_{X/Y} \rightarrow TY$. Such a lifting ρ and any section σ of $Y_{\text{conn}}|_X$ together determine a section Ξ of $\Psi_{X/Y}$ via κ , and hence an isomorphism between graded algebras

$$\exp_{X/Y,\Xi}^* : \mathcal{A}^\bullet(X_Y^{(\infty)}) \xrightarrow{\cong} \mathcal{A}_X^{0,\bullet}(\hat{\mathcal{S}}^\bullet(N_{X/Y}^\vee))$$

The rest of the job is to determine which differential we should put on the codomain in terms of ρ and σ to make it into an isomorphism of dgas.

As in Theorem 3.7, denote the graph of i by

$$\tilde{i} := (\text{Id}, i) : X \rightarrow X \times Y.$$

Think of X as a submanifold of $X \times Y$, by Theorem 3.7 a section σ of $Y_{\text{conn}}|_X$

$$\exp_\sigma^* : (\mathcal{A}^\bullet(X_{X \times Y}^{(\infty)}), \bar{\partial}) \xrightarrow{\cong} (\mathcal{A}_X^{0,\bullet}(\hat{\mathcal{S}}^\bullet(T^*Y)), D_\sigma)$$

where T^*Y is understood as the pullback i^*T^*Y (we will omit i^* whenever the meaning of the notation is clear from the context). The derivation

$$D_\sigma = \bar{\partial} + \sum_{n \geq 2} \tilde{R}_n \tag{4.5}$$

and \tilde{R}_n is induced by (pullback of) the covariant derivatives of curvature forms on Y .

Note we have the commutative diagram of holomorphic maps

$$\begin{array}{ccc} X & \xrightarrow{\tilde{i}} & X \times Y \\ \parallel & & \downarrow \pi \\ X & \xrightarrow{i} & Y \end{array}$$

where $\pi : X \times Y \rightarrow Y$ is the natural projection. Thus by functoriality, π induces an injective homomorphism of dgas

$$\pi^* : (\mathcal{A}^\bullet(X_Y^{(\infty)}), \bar{\partial}) \rightarrow (\mathcal{A}^{0,\bullet}(X_{X \times Y}), \bar{\partial}), \quad [\eta]_\infty \mapsto [\pi^* \eta]_\infty. \quad (4.6)$$

We then compose π^* with the isomorphism \exp_σ^* to get

$$\exp_\sigma^* \circ \pi^* : (\mathcal{A}^\bullet(X_Y^{(\infty)}), \bar{\partial}) \rightarrow (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y), D_\sigma))$$

We then extend ρ^\vee to obtain a homomorphism of graded algebras

$$\rho^\vee : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y)) \rightarrow \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee)). \quad (4.7)$$

Compose with $\exp_\sigma^* \circ \pi^*$ in (4.11), we get a homomorphism of graded algebras

$$\rho^\vee \circ \exp_\sigma^* \circ \pi^* : \mathcal{A}^\bullet(X_Y^{(\infty)}) \rightarrow \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee))$$

which, in fact, coincides with $\exp_{X/Y,\Xi}^*$.

Lemma 4.1. *With all the notations above, we have*

$$\exp_{X/Y,\Xi}^* = \rho^\vee \circ \exp_\sigma^* \circ \pi^* \quad (4.8)$$

Proof. Follows immediately from the definition of κ . \square

Via the isomorphism $\exp_{X/Y,\Xi}^*$ we can transfer the $\bar{\partial}$ -derivation on $\mathcal{A}^\bullet(X_Y^{(\infty)})$ to one on $\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee))$, denoted as \mathfrak{D} , i.e.,

$$\mathfrak{D} = \exp_{X/Y,\Xi}^* \circ \bar{\partial} \circ (\exp_{X/Y,\Xi}^*)^{-1}.$$

Hence $\exp_{X/Y,\Xi}^*$ becomes an isomorphism of dgas

$$\exp_{X/Y,\Xi}^* : (\mathcal{A}^\bullet(X_Y^{(\infty)}), \bar{\partial}) \xrightarrow{\cong} (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee)), \mathfrak{D}).$$

Thus we can also transfer the homomorphism π^* in (4.6) to a homomorphism

$$\tilde{\pi}^* : (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee)), \mathcal{D}) \rightarrow (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y), D_\sigma),$$

that is, let

$$\tilde{\pi}^* = \exp_\sigma^* \circ \pi^* \circ (\exp_{X/Y, \Xi}^*)^{-1} \quad (4.9)$$

Then we have the following commutative diagram:

$$\begin{array}{ccc} (\mathcal{A}^\bullet(X_Y^{(\infty)}), \bar{\partial}) & \xrightarrow{\pi^*} & (\mathcal{A}^{0,\bullet}(X_{X \times Y}^{(\infty)}), \bar{\partial}) \\ \cong \downarrow \exp_{X/Y, \Xi}^* & & \cong \downarrow \exp_\sigma^* \\ (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee)), \mathcal{D}) & \xrightarrow{\tilde{\pi}^*} & (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y), D) \end{array}$$

Note that by (4.8) and (4.9) we have

$$\rho^\vee \circ \tilde{\pi}^* = \text{Id} : (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee)), \mathcal{D}) \rightarrow (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee)), \mathcal{D}) \quad (4.10)$$

even though ρ^\vee is not a homomorphism of dgas.

4.2.2. *Description of $\tilde{\pi}^*$.* From now on, let us assume Y is Kähler and the section σ of $Y_{\text{conn}}|_X$ is determined by the associated $(1, 0)$ -connection ∇ . Yet we want to keep the reader aware that all the arguments and computations below will still work even if we drop the Kähler condition and talk about arbitrary section σ . The reason for the Kähler assumption is that those terms in the final formula (4.20) will then have clearer geometric meanings.

By the discussion at the end of §3.3, the homomorphism

$$\exp_\sigma^* \circ \pi^* : (\mathcal{A}^\bullet(X_Y^{(\infty)}), \bar{\partial}) \rightarrow (\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y), D_\sigma)$$

looks like

$$[\eta]_\infty \mapsto (i^*\eta, i^*\nabla\eta, i^*\nabla^2\eta, \dots) \quad (4.11)$$

since $\tilde{i}^*\nabla^n\pi^*\eta = i^*\nabla^n\eta$, where the ∇ on the RHS is the original $(1, 0)$ -connection on Y while the one on the LHS is the pullback one along Y -fibers of $X \times Y$.

To give a (more or less) concrete description of the homomorphism $\tilde{\pi}^*$, let us first make some conventions on notations. We abuse the notations and write $TY = TX \oplus N_{X/Y}$ and $T^*Y = T^*X \oplus N_{X/Y}^\vee$ induced by the fixed splittings (we omit i^* from now on). The decompositions extend to tensor products, i.e., tensor product $TY^{\otimes n}$ can be decomposed into direct sum of mixed tensors of TX and $N_{X/Y}$ components and similarly for $T^*Y^{\otimes n}$. The same for symmetric tensor products:

$$S^n TY = \bigoplus_{p+q=n} S^p TX \cdot S^q N_{X/Y} \quad \text{and} \quad S^n T^*Y = \bigoplus_{p+q=n} S^p T^*X \cdot S^q N_{X/Y}^\vee, \quad (4.12)$$

where the dot stands for the commutative multiplication in the commutative algebras.

We define a derivation

$$\overline{\nabla} : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet T^*Y) \rightarrow \mathcal{A}_X^{0,\bullet}(\hat{S}^{\bullet+1} T^*Y) \quad (4.13)$$

of degree 0 with respect to the grading from $\mathcal{A}_X^{0,\bullet}$ as the composition of the operators

$$\overline{\nabla} := \overline{\text{Sym}} \circ \nabla$$

Namely, for any $\eta \in \mathcal{A}_X^{0,\bullet}(S^n T^*Y)$, first apply the (induced) connection ∇ to get a $T^*X \otimes S^n T^*Y$ -valued form

$$\nabla \eta \in \mathcal{A}_X^{0,\bullet}(T^*X \otimes S^n T^*Y),$$

then apply a variation of the usual symmetrization map

$$\overline{\text{Sym}} : T^*X \otimes S^n T^*Y \rightarrow S^{n+1} T^*Y,$$

which we define on each component of the decomposition (4.12) as

$$\overline{\text{Sym}}_{m,n} : T^*X \otimes (S^{m-1} T^*X \cdot S^{n-m+1} N_{X/Y}^\vee) \rightarrow S^m T^*X \cdot S^{n-m+1} N_{X/Y}^\vee$$

by the formula

$$\begin{aligned} \overline{\text{Sym}}_{m,n}(v_0 \otimes (v_1 \cdot v_2 \cdots v_n)) &= \frac{1}{m} v_0 \cdot v_1 \cdots v_n, \\ \forall v_0 \otimes (v_1 \cdot v_2 \cdots v_n) &\in T^*X \otimes (S^{m-1} T^*X \cdot S^{n-m+1} N_{X/Y}^\vee) \end{aligned} \quad (4.14)$$

and we finally get

$$\bar{\nabla}\eta \in \mathcal{A}_X^{0,\bullet}(S^{n+1}T^*Y)$$

When $\eta \in \mathcal{A}_X^{0,\bullet}(S^0(T^*Y)) = \mathcal{A}_X^{0,\bullet}$, $\bar{\nabla}\eta = \tau^\vee(\partial\eta) \in \mathcal{A}_X^{0,\bullet}(T^*Y)$ where ∂ is the $(1,0)$ -derivation on X . We can inductively apply $\bar{\nabla}$ and get

$$\bar{\nabla}^k : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y)) \rightarrow \mathcal{A}_X^{0,\bullet}(\hat{S}^{\bullet+k}(T^*Y))$$

One trivial remark is that, since $N_{X/Y}^\vee$ is naturally identified as a subbundle of T^*Y via $p^\vee : N_{X/Y}^\vee \rightarrow T^*Y$, we can form the restriction of $\bar{\nabla}^k$

$$\bar{\nabla}^k : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee)) \rightarrow \mathcal{A}_X^{0,\bullet}(\hat{S}^{\bullet+k}(T^*Y))$$

Note here $\bar{\nabla}^0$ is the natural inclusion $\hat{S}^\bullet(N_{X/Y}^\vee) \hookrightarrow \hat{S}^\bullet(T^*Y)$.

Proposition 4.2. *We have*

$$\tilde{\pi}^* = \sum_{k=0}^{\infty} \bar{\nabla}^k$$

where $\tilde{\pi}^* : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee)) \rightarrow \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y))$ is as in (4.9). That is, given $\mu = (\mu_k)_{k=0}^\infty \in \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee))$, the n -th component of its image $\nu = \tilde{\pi}^*(\mu)$ is

$$\nu_n = \sum_{k=0}^n \bar{\nabla}^k \mu_{n-k}.$$

Proof. Assume that $\exp_{X/Y,\Xi}^*([\eta]_\infty) = \mu$ where $[\eta]_\infty \in \mathcal{A}^\bullet(X_Y^{(\infty)})$. By (4.8) and (4.11), this means

$$\mu_k = \rho^\vee(i^*\nabla^k\eta)$$

where ρ^\vee is the projection

$$\rho^\vee : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y)) \rightarrow \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee))$$

as in (4.7). Moreover, by the definition of $\tilde{\pi}^*$ (4.9), we have

$$\tilde{\pi}^*(\mu) = \exp_\sigma^* \circ \pi^* \circ (\exp_{X/Y,\Xi}^*)^{-1}(\mu) = \exp_\sigma^* \circ \pi^*([\eta]_\infty) = (i^*\nabla^k\eta)_{k=0}^\infty.$$

Thus all we need to show is that

$$i^*\nabla^n\eta = \sum_{k=0}^n \bar{\nabla}^k(\rho^\vee(i^*\nabla^{n-k}\eta))$$

for all $n \geq 0$. We prove it by induction on n . The $n = 0$ case is trivial. For $n \geq 1$, note we can write

$$i^*\nabla^n\eta = \rho^\vee(i^*\nabla^n\eta) + \text{complement in } T^*X \cdot S^{n-1}T^*Y \subset S^nT^*Y$$

via the decompositions (4.12). The second term on the right hand side is nothing but $\bar{\nabla}(i^*\nabla^{n-1}\eta)$. To see this, we look at what happens when one evaluate $i^*\nabla^n\eta$ against some section s of $TY^{\otimes n}$, which lies in a mixed tensor of m copies of TX ($m \geq 1$) and $n - m$ copies of N (of arbitrary order). One can shift one of the TX -factors to the first place and plug it into the first ∇ in $i^*\nabla^n\eta$, because of the symmetry of $i^*\nabla^n\eta$. This implies that $i^*\nabla^n\eta$ should be a symmetrization of $\nabla(i^*\nabla^{n-1}\eta)$ (in which the very first ∇ stands again for the induced connection on i^*TY). The value of latter on s , however, is m times what we need since s contains m TX -factors. This explains the fractional factor $1/m$ in the formula (4.14). Finally we end the proof by applying the inductive assumption. \square

Remark 4.3. One can further expand $\bar{\nabla}$ and $\bar{\nabla}^k$ in terms of second fundamental form and shape operator, which might be interesting. As we will see immediately, however, only a small part of the expansions will be relevant to the derivation \mathfrak{D} , so we postpone the general discussion for the future.

4.2.3. *Description of the derivation \mathfrak{D} .* To determine the derivation \mathfrak{D} , note that by (4.10) we have

$$\mathfrak{D} = \rho^\vee \circ \tilde{\pi}^* \circ \mathfrak{D} = \rho^\vee \circ D \circ \tilde{\pi}^*$$

where the last equality is by the definition of \mathfrak{D} . Thus for given $\mu = (\mu_k)_{k=0}^\infty \in \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee))$, the n -th component of $\mathfrak{D}\mu$ is

$$(\mathfrak{D}\mu)_n = \sum_{s+t=n} \rho^\vee \circ \bar{\partial} \circ \bar{\nabla}^s \mu_t + \sum_{\substack{r+s+t=n \\ r \geq 1}} \rho^\vee \circ \tilde{R}_{r+1} \circ \bar{\nabla}^s \mu_t \quad (4.15)$$

by Proposition 4.2 and (4.5). To simplify the right hand side further, first observe that all we need are the components of $\bar{\nabla}^s \mu_t$ lying in $\hat{S}^\bullet(N_{X/Y}^\vee)$ and $T^*X \cdot \hat{S}^\bullet(N_{X/Y}^\vee)$ and we can ignore the remaining ones in $S^2T^*X \cdot \hat{S}^\bullet(T^*Y)$.

The reason is that if we apply the derivations $\bar{\partial}$ and \tilde{R}_n on something from $S^2T^*X \cdot \hat{S}^\bullet(T^*Y)$, the result must lie in $T^*X \cdot \hat{S}^\bullet(T^*Y)$, which will be eliminated by the projection ρ^\vee .

Thus we first denote the projections onto the only two 'effective' components respectively by

$$P_0 = p^\vee \circ \rho^\vee : \hat{S}^\bullet(T^*Y) \rightarrow \hat{S}^\bullet(N_{X/Y}^\vee) \subset \hat{S}^\bullet(T^*Y)$$

and

$$P_1 : \hat{S}^\bullet(T^*Y) \rightarrow T^*X \cdot \hat{S}^\bullet(N_{X/Y}^\vee) \subset \hat{S}^\bullet(T^*Y).$$

Secondly, we define the derivation of degree +1

$$\tilde{\beta}_{X/Y} : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y)) \rightarrow \mathcal{A}_X^{0,\bullet+1}(\hat{S}^\bullet(T^*Y))$$

induced by

$$\beta_{X/Y} \in \mathcal{A}_X^{0,1}(\text{Hom}(T^*X, N_{X/Y}^\vee)) \subset \mathcal{A}_X^{0,1}(\text{Hom}(T^*Y, T^*Y)).$$

where the last inclusion comes again from the splitting. Note that $\tilde{\beta}_{X/Y}$ acts on $\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee)) \subset \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y))$ as the zero map. So we can also think of $\tilde{\beta}_{X/Y}$ as the operator

$$\tilde{\beta}_{X/Y} : \mathcal{A}_X^{0,\bullet}(T^*X \cdot \hat{S}^\bullet(N_{X/Y}^\vee)) \rightarrow \mathcal{A}_X^{0,\bullet+1}(\hat{S}^\bullet(N_{X/Y}^\vee))$$

The following lemma is immediate from (4.3)

Lemma 4.4. *As derivations $\mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(T^*Y)) \rightarrow \mathcal{A}_X^{0,\bullet+1}(\hat{S}^\bullet(N_{X/Y}^\vee))$*

$$[\rho^\vee, \bar{\partial}] = \tilde{\beta}_{X/Y} \circ P_1$$

Hence the first term on RHS of (??) can be rewritten as

$$\begin{aligned} \sum_{s+t=n} \rho^\vee \circ \bar{\partial} \circ \bar{\nabla}^s \mu_t &= \sum_{s+t=n} \bar{\partial} \circ \rho^\vee \circ \bar{\nabla}^s \mu_t + \sum_{s+t=n} \tilde{\beta}_X \circ P_1 \circ \bar{\nabla}^s \mu_t \\ &= \bar{\partial} \mu_n + \sum_{s+t=n} \tilde{\beta}_X \circ P_1 \circ \bar{\nabla}^s \mu_t \end{aligned} \tag{4.16}$$

The last equality is because that $\bar{\nabla}^s \mu_t \in \mathcal{A}_X^{0,\bullet}(T^*X \cdot \hat{S}^\bullet(T^*Y))$ unless $s = 0$.

Thirdly, we split $R_n \in \mathcal{A}_X^{0,1}(\text{Hom}(T^*Y, S^n T^*Y))$ into two components,

$$R_n^\perp := \rho^\vee \circ R_n \circ \rho^\vee \in \mathcal{A}_X^{0,1}(\text{Hom}(N_{X/Y}^\vee, S^n N_{X/Y}^\vee))$$

and

$$R_n^\top := \rho^\vee \circ R_n \circ \tau^\vee \in \mathcal{A}_X^{0,1}(\text{Hom}(T^*X, S^n N_{X/Y}^\vee))$$

and denote the induced operators by

$$\tilde{R}_n^\perp : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee)) \rightarrow \mathcal{A}_X^{0,\bullet+1}(\hat{S}^{\bullet+n-1}(N_{X/Y}^\vee))$$

and

$$\tilde{R}_n^\top : \mathcal{A}_X^{0,\bullet}(T^*X \cdot \hat{S}^\bullet(N_{X/Y}^\vee)) \rightarrow \mathcal{A}_X^{0,\bullet+1}(\hat{S}^{\bullet+n}(N_{X/Y}^\vee))$$

respectively. To unify notations, we also write $R_1^\top := \beta_{X/Y}$ and $\tilde{R}_1^\top := \tilde{\beta}_{X/Y}$.

We can then split the second term on RHS of (4.15):

$$\begin{aligned} & \sum_{\substack{r+s+t=n \\ r \geq 1}} \rho^\vee \circ \tilde{R}_{r+1} \circ \bar{\nabla}^s \mu_t \\ &= \sum_{\substack{r+s+t=n \\ r \geq 1}} \rho^\vee \circ \tilde{R}_{r+1} \circ P_0 \circ \bar{\nabla}^s \mu_t + \sum_{\substack{r+s+t=n \\ r \geq 1}} \rho^\vee \circ \tilde{R}_{r+1} \circ P_1 \circ \bar{\nabla}^s \mu_t \\ &= \sum_{\substack{r+s+t=n \\ r \geq 1}} \tilde{R}_{r+1}^\perp \circ P_0 \circ \bar{\nabla}^s \mu_t + \sum_{\substack{r+s+t=n \\ r \geq 1}} \tilde{R}_{r+1}^\top \circ P_1 \circ \bar{\nabla}^s \mu_t \\ &= \sum_{k=2}^n \tilde{R}_k^\perp \circ \mu_{n-k+1} + \sum_{\substack{r+s+t=n \\ r \geq 1}} \tilde{R}_{r+1}^\top \circ P_1 \circ \bar{\nabla}^s \mu_t \end{aligned} \tag{4.17}$$

Combine (4.15), (4.16) and (4.17) we get

$$(\mathcal{D}\mu)_n = \bar{\partial}\mu_n + \sum_{k=2}^n \tilde{R}_k^\perp \circ \mu_{n-k+1} + \sum_{\substack{r+s+t=n \\ r,t \geq 0, s \geq 1}} \tilde{R}_{r+1}^\top \circ P_1 \circ \bar{\nabla}^s \mu_t \tag{4.18}$$

Finally, to compute the terms $P_1 \circ \bar{\nabla}^s \mu_t$, we define two derivations of degree 0 with respect to the grading from $\mathcal{A}_X^{0,\bullet}$:

$$\bar{\nabla}^\perp : \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee)) \rightarrow \mathcal{A}_X^{0,\bullet}(T^*X \cdot \hat{S}^\bullet(N_{X/Y}^\vee))$$

induced by the connection ∇^\perp on $N_{X/Y}^\vee$ the same way as we define $\bar{\nabla}$ in (4.13). $\nabla^\perp f$ of a function f is again understood as ∂f , the $(1,0)$ differential.

The second one

$$\tilde{A} : \mathcal{A}_X^{0,\bullet}(T^*X \cdot \hat{S}^\bullet(N_{X/Y}^\vee)) \rightarrow \mathcal{A}_X^{0,\bullet}(T^*X \cdot \hat{S}^{\bullet+1}(N_{X/Y}^\vee))$$

induced by the shape operator $A : T^*X \rightarrow T^*X \otimes N_{X/Y}^\vee$ as in (4.4) yet again with the images symmetrized.

Lemma 4.5. *With the notations above, we have*

$$P_1 \circ \bar{\nabla} = \bar{\nabla}^\perp \circ P_0 + \tilde{A} \circ P_1,$$

thus $\forall s \geq 1, \forall \mu \in \hat{S}^\bullet(N_{X/Y}^\vee)$,

$$P_1 \circ \bar{\nabla}^s \mu = (\tilde{A})^{s-1} \circ \bar{\nabla}^\perp \mu. \quad (4.19)$$

Applying the equality (4.19) to (4.18), we eventually get

Theorem 4.6. *Given $\mu = (\mu_k)_{k=0}^\infty \in \mathcal{A}_X^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee))$, the n -th component of $\mathfrak{D}\mu$ is*

$$(\mathfrak{D}\mu)_n = \bar{\partial}\mu_n + \sum_{k=2}^n \tilde{R}_k^\perp \circ \mu_{n-k+1} + \sum_{\substack{r+s+t=n \\ r,t \geq 0, s \geq 1}} \tilde{R}_{r+1}^\top \circ \tilde{A}^{s-1} \circ \bar{\nabla}^\perp \mu_t.$$

In other words,

$$\mathfrak{D} = \bar{\partial} + \sum_{k \geq 2} \tilde{R}_k^\perp + \sum_{p \geq 1, q \geq 0} \tilde{R}_p^\top \circ \tilde{A}^q \circ \bar{\nabla}^\perp. \quad (4.20)$$

Remark 4.7. From Theorem 4.6 we see that, even when \mathfrak{D} acts on a function f (or a form), higher term would be produced in general. Namely, by (4.20)

$$\mathfrak{D}f = \bar{\partial}f + \sum_{p \geq 1, q \geq 0} \tilde{R}_p^\top \circ \tilde{A}^q \circ \partial f \quad (4.21)$$

This is a huge difference between the general situation and the case of diagonal embedding.

Remark 4.8. Although we get $\mathfrak{D}^2 = 0$ for free from how we construct it, it is still an interesting (yet tedious) exercise to verify it by hands and one will observe the Gauss-Codazzi-Ricci equations in classical differential geometry. This will be included in the Appendix.

5. L_∞ -ALGEBROIDS

In this section we will show that there exists a L_∞ -algebroid structure on the complex $\mathcal{A}_X^{0,\bullet}(\mathbb{N}_{X/Y}[-1]) = \mathcal{A}_X^{0,\bullet-1}(\mathbb{N}_{X/Y})$ whose Chevalley-Eilenberg complex is exactly the Dolbeault dga $\mathcal{A}^\bullet(Y_X^{(\infty)})$. Since throughout this section the background algebra is the Dolbeault dga $(\mathcal{A}^{0,\bullet}(X), \bar{\partial})$ of the submanifold X , we will just write it as $\mathcal{A}^{0,\bullet}$.

Definition 5.1. An L_∞ -algebroid or an *strongly homotopy Lie algebroid* over the dga $\mathcal{A}^{0,\bullet}$ is the following data:

- (1) A dg-module (L^\bullet, d) over the dga $\mathcal{A}^{0,\bullet}$ which is an L_∞ -algebra, i.e., equipped with a series of (graded) antisymmetric \mathbb{C} -linear n -ary operations

$$l_n : (L^\bullet)^{\otimes n} \rightarrow L^{\bullet+2-n}, \quad x_1 \otimes \cdots \otimes x_n \mapsto [x_1, \dots, x_n]_n, \quad n \geq 2$$

(Note here tensors are over \mathbb{C} !) with $\deg(l_n) = 2 - n$, satisfying the usual (generalized) Jacobi identities. If we define $l_1 := d$, then the Jacobi identities look like

$$\sum_{i+j=n+1} \sum_{\sigma \in \text{Sh}(i, n-i)} \pm l_j(l_i(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(i)}) \otimes \cdots \otimes x_{\sigma(n)}) = 0 \quad (5.1)$$

- (2) An *anchor map* α which is a L_∞ -morphism from L^\bullet to the DGLA $(\mathcal{A}^{0,\bullet}(TX), [,], \bar{\partial})$ with the natural Lie bracket of tangent vector fields, i.e., there is a collection of antisymmetric maps of degree 0

$$\alpha_n : (L^\bullet)^{\otimes n} \rightarrow \mathcal{A}^{0,\bullet-n+1}(TX), \quad n \geq 1$$

compatible with the L_∞ -structures. Moreover, α_n is required to be $\mathcal{A}^{0,\bullet}$ -linear.

- (3) Each n -bracket l_n is an $\mathcal{A}^{0,\bullet}$ -derivation on each of its arguments via the anchor map α , that is,

$$[x_1, \dots, x_{n-1}, \omega \cdot x_n]_n = (\mathcal{L}_{\alpha_{n-1}(x_1, \dots, x_{n-1})} \omega) \cdot x_n \pm \omega \cdot [x_1, \dots, x_n]_n \quad (5.2)$$

where \mathcal{L} stands for Lie derivatives.

Assume that L^\bullet satisfies some finiteness property, e.g., finitely generated and projective over $\mathcal{A}^{0,\bullet}$. Then we can consider the Chevalley-Eilenberg complex of the L_∞ -algebroid L^\bullet , i.e., the completed symmetric algebra $\widehat{S}(L^{\vee\bullet}[-1])$, where

$$L^{\vee\bullet} = \text{Hom}_{\mathcal{A}^{0,\bullet}}^\bullet(L^\bullet, \mathcal{A}^{0,\bullet})$$

is the dual $\mathcal{A}^{0,\bullet}$ -module of L^\bullet , equipped with the natural differential. Each bracket l_n ($n \geq 2$) gives a map $l_n^* : L^{\vee\bullet}[-1] \rightarrow S^n(L^{\vee\bullet}[-1])$ of degree 1 by the formula

$$l_n^*(\eta)(x_1, \dots, x_n) = \sum_{i=1}^n \pm \mathcal{L}_{\alpha_{n-1}(x_1, \dots, \hat{x}_i, \dots, x_n)} \eta(x_i) - (-1)^{|n|} \eta([x_1, \dots, x_n]) \quad (5.3)$$

We can then extend l_n^* uniquely to a derivation d_n of degree 1 of the algebra $\widehat{S}(L^{\vee\bullet}[-1])$ such that d_n restricted on $S^0(L^{\vee\bullet}) = \mathcal{A}^{0,\bullet}$ is a map $d_n : \mathcal{A}^{0,\bullet} \rightarrow S^{n-1}(L^{\vee\bullet}[-1])$ defined by

$$d_n(\omega)(x_1, \dots, x_{n-1}) = \mathcal{L}_{\alpha_{n-1}(x_1, \dots, x_{n-1})} \omega \quad (5.4)$$

Put d_n 's all together we get a differential on $\widehat{S}(L^{\vee\bullet}[-1])$

$$D = d + \sum_{n \geq 2} d_n$$

and all conditions in Definition (5.1) can be expressed as one condition

$$D^2 = 0$$

In other words, $(\widehat{S}(L^{\vee\bullet}[-1]), D)$ is a dga. Conversely, any derivation D on $\widehat{S}(L^{\vee\bullet}[-1])$ of degree 1 satisfying $D^2 = 0$ and respecting the natural filtration on $\widehat{S}(L^{\vee\bullet}[-1])$ determines a L_∞ -algebroid structure on L^\bullet such that the brackets and anchor maps are determined by the formulas (5.3) and (5.4).

Thus we can repackage the differential \mathfrak{D} in the formula (4.20) in the language of L_∞ -algebroids. First of all, the underlying $\mathcal{A}^{0,\bullet}$ -module is

$$(L^\bullet, d) = (\mathcal{A}^{0,\bullet}(N_{X/Y}[-1]), \bar{d}) = (\mathcal{A}^{0,\bullet-1}(N_{X/Y}), \bar{d})$$

Next, by comparing (4.21) with (5.4) we see that the anchor maps should be give by the recursive formulas:

$$\alpha_1 = R_1^\top = \beta_{X/Y} \in \mathcal{A}^{0,1}(\text{Hom}(N_{X/Y}, TX)), \quad (5.5)$$

$$\alpha_n = R_n^\top + \sum_{\sigma \in \text{Sh}(n-1,1)} \pm A \circ (\alpha_{n-1} \otimes 1) \circ \sigma \in \mathcal{A}^{0,1}(\text{Hom}(S^n N[-1], TX)) \quad n \geq 2 \quad (5.6)$$

(Recall that the shape operator $A : TX \otimes N_{X/Y} \rightarrow TX$ has TX as its first slot and $N_{X/Y}$ as the second.)

Finally, the brackets can be extracted from (4.20) and the formulas for α_n 's by using (5.4):

$$\mathfrak{L}_n = R_n^\perp + \sum_{\sigma \in \text{Sh}(n-1,1)} \pm \nabla^\perp \circ (\alpha_{n-1} \otimes 1) \circ \sigma, \quad n \geq 2 \quad (5.7)$$

where the tensor is over \mathbb{C} and the connection ∇^\perp on the normal bundle is considered as a map

$$\nabla^\perp : \mathcal{A}^{0,\bullet}(TX) \otimes_{\mathbb{C}} \mathcal{A}^{0,\bullet}(N_{X/Y}[-1]) \rightarrow \mathcal{A}^{0,\bullet}(N_{X/Y}[-1])$$

of degree zero.

Remark 5.2. One can also interpret the homomorphism

$$\tilde{\pi}^* : (\mathcal{A}^{0,\bullet}(\hat{S}^\bullet(N_{X/Y}^\vee)), \mathfrak{D}) \rightarrow (\mathcal{A}^{0,\bullet}(\hat{S}^\bullet(T^*Y), D)$$

defined in (4.9) as an L_∞ -morphism from $\mathcal{A}^{0,\bullet}(TY[-1])$, thought of as an L_∞ -algebroid with zero anchor map, to $\mathcal{A}^{0,\bullet}(N_{X/Y}[-1])$. But note that this morphism commutes with anchor maps only up to homotopy.

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