NOTES ON FORMAL NEIGHBORHOODS AND JET BUNDLES

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ABSTRACT. The purpose of this note is to review the construction of smooth and holomorphic jet bundles and its relation to formal neighborhood of the diagonal embedding. I will show that there is a natural notion of "Dolbeault dgas" which works for formal neighborhoods of arbitrary analytical embeddings. An algebraic proof of a theorem by M. Kapranov will be addressed at the end regarding the structure of such dga in the case of diagonal embedding.

CONTENTS

1. C^∞-JET BUNDLES

The notion of C^∞-jet bundle provides an appropriate place where one can talk about Taylor expansions (or jets) of smooth functions on a smooth manifold. Let X be a manifold and $p \in X$ a point, for each nonnegative integer r, we define the algebra J_p^r to be the quotient of $C^\infty(X)$ by the ideal

$$I_p^{(r)} = \{\text{functions whose derivatives up to order } r \text{ all vanish at } p\}.$$

In fact, if $\mathfrak{m}_p$ denotes the maximal ideal of the commutative algebra $C^\infty(X)$ consisting of functions vanishing at p, we have

$$I_p^{(r)} = \mathfrak{m}_p^{r+1}.$$

The **Taylor expansion of order** r (or r**-jet**) of a function f at p is defined to be the equivalence class

$$j_p^r f := [f]_p^r \in J_p^r = C^\infty(X)/\mathfrak{m}_p^{r+1}$$

Key words and phrases. ...

Note that J_p^r is determined by local data around p, so in fact we should use the algebra $C^\infty(X)_p$ of germs of smooth functions at p instead of $C^\infty(X)$. But this algebra itself is a quotient of $C^\infty(X)$, so everything is fine. Indeed, let $\mathfrak{n}_p$ be the ideal of all global smooth functions vanishing at some neighborhood of p, then there is a natural isomorphism

$$C^\infty(X)/\mathfrak{n}_p \cong C^\infty(X)_p.$$

This is not true in the holomorphic world.

If we choose a local coordinate chart (U, x_i) containing $p = (p_1, \ldots, p_n)$, then there is an isomorphism

$$J_p^r \cong \mathbb{C}[x_1, \ldots, x_n]/(x_1 - p_1, \ldots, x_n - p_n)^{r+1}$$
$$[f]_p^r \mapsto \sum_{|I| \le r} a_{i_1,\ldots,i_n}(x_1 - p_1)^{i_1} \cdots (x_n - p_n)^{i_n}$$

where

$$a_{i_1,\ldots,i_n} = \frac{1}{i_1! \ldots i_n!} \frac{\partial^{i_1+\cdots+i_n} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}(p)$$

Consider the above construction for any r, we have an inverse system

$$\mathbb{C} = J_p^0 \leftarrow J_p^1 \leftarrow J_p^1 \leftarrow \cdots$$

with connecting maps being the natural quotient maps. So we can define the space of ∞-jets

$$J_p^\infty := \varprojlim_r J_p^r$$

which is of infinite dimension. Moreover, the above isomorphisms for J_p^r are compatible with the inverse system,so under the local chart (U, x_i) we have an isomorphism

$$J_p^\infty \cong \varprojlim_r \mathbb{C}[x_1, \ldots, x_n]/(x_1 - p_1, \ldots, x_n - p_n)^{r+1} = \mathbb{C}[\![x_1 - p_1, \ldots, x_n - p_n]\!].$$

Remark 1.1. By definition of inverse limit, we have the natural 'Taylor expansion' map

$$j_p : C^\infty(X) \to J_p^\infty.$$

A result by E. Borel ([1]) says that this map is surjective. In other words, there always exists (locally) a C^∞-function with the given Taylor expansion. Again this fails when it comes to holomorphic functions.The kernel of j_p is

$$I_p^{(\infty)} = \bigcap_{r=0}^{\infty} I_p^{(r)} = \bigcap_{r=0}^{\infty} \mathfrak{m}_p^{r+1}.$$

Next I will define the jet bundles J^r (resp. J^∞) whose fibres are J^r_p (resp. J^∞_p). For this purpose, I only need to tell you the sheaf of smooth sections of the bundles. Let us assume that $\dim X = 1$ for simplicity. For any open subset $V \subset X$, among the giant garbage of all 'discrete' sections

$$s : V \to \bigsqcup_{p \in V} J^r_p, \quad s(p) \in J^r_p, \ \forall\, p \in V,$$

I only look at those with the following property: for any $y \in V$, there exists a local chart (U, x) containing y, such that, under the isomorphisms $J^r_p = \mathbb{C}[x]/(x-p)^{r+1}$,

$$s(p) = [a_0(p) + a_1(p)(x-p) + \cdots + a_r(p)(x-p)^r] \in J^r_p$$

where $a_0, \dots, a_r \in C^\infty(U)$. We denote the set of all such sections by $C^\infty(V, J^r)$. It has a $C^\infty(U)$-module structure defined by pointwise addition and multiplication, and moreover is a commutative algebra. One can show that this really defines a sheaf over X and provides a realization of J^r as a smooth vector bundle. The only thing need to be checked is that the condition above is independent of the choice of local charts, which is merely an exercise using the chain rule. In fact, from the definition we get a trivialization of J^r on any local chart (U, x) with the frame

$$[1], [x-p], [(x-p)^2], \dots, [(x-p)^r].$$

In a similar way one can define J^∞, or just by $J^\infty = \varprojlim J^r$.

A section of the jet bundles does not necessarily come from an actual smooth function. In fact, the general sections are much more than those induced by functions. Nevertheless, we have canonical maps of algebras

$$j^r : C^\infty(X) \to C^\infty(X, J^r).$$

and

$$j^\infty : C^\infty(X) \to C^\infty(X, J^\infty)$$

under which the images of a function are called prolongations. However, j^r is not a $C^\infty(X)$-linear map. It is a differential operator of order r.

From the description above of local sections of the jet bundles, one can see there are actually two variables x and p involved, though x is a formal variable. This inspires us to think of sections of jet bundles as 'formal functions' on $X \times X$ near the diagonal. This is the first step towards the general concept of formal neighborhood. Let's see how it works.

We write $X \times X$ as $X' \times X''$ to distinguish the two factors and denote the corresponding projections by

$$pr' : X' \times X'' \to X' \quad \text{and} \quad pr'' : X' \times X'' \to X''$$

respectively. There is a natural splitting of the tangent bundle

$$T(X' \times X'') = pr'^{*}TX' \oplus pr''^{*}TX''$$

or in shorthanded notation

$$T(X' \times X'') = TX' \oplus TX''$$

The diagonal map $\Delta : X \hookrightarrow X' \times X''$ embeds X as a closed submanifold, whose image we also write as Δ by abuse of notation. A local chart (U, x) of X gives rise to a chart $(U' \times U'', x', x'')$ of $X' \times X''$, with $x' = x \otimes 1$ and $x'' = 1 \otimes x$ as coordinate functions. Then one can 'realize' a local section of J^r on U as a function of the form

$$f(x', x'') = a_0(x') + a_1(x')(x'' - x') + \cdots + a_r(x')(x'' - x')^r$$

on $U' \times U''$. More precisely, consider the closed ideal of the algebra $C^\infty(X' \times X'')$:

$$I_\Delta^{(r)} := \left\{ f \in C^\infty(X' \times X'') \;\middle|\; \begin{array}{c} V_1 V_2 \cdots V_l f|_\Delta = 0,\ \forall\, V_j \in C^\infty(TX''), \\ \forall\, 1 \le j \le l,\ 0 \le l \le r. \end{array} \right\}. \tag{1}$$

In other words, it consists of functions on $X' \times X''$ whose derivatives in X''-directions restricted on the diagonal vanish up to r-th order. The algebra of smooth functions on the r-th order formal neighborhood of the diagonal then can be defined as

$$C^\infty(X_\Delta^{(r)}) := C^\infty(X' \times X'')/I_\Delta^{(r)}.$$

Note that any function defined only on an open neighborhood of the diagonal also gives an equivalence class in $C^\infty(X_\Delta^{(r)})$, but by multiplying with a bump function near the diagonal one can still take a function globally define on $X' \times X''$ as a representative of the same class.

There is an obvious isomorphism

$$\tau_1 : C^\infty(X_\Delta^{(r)}) \xrightarrow{\cong} C^\infty(X, J^r), \quad [f] \mapsto s_f, \tag{2}$$

where s_f is defined by

$$s_f(p) = j_p^r(f|_{\{p\} \times X''}),$$

that is, we restrict f on the fibre $\{p\} \times X''$, identified naturally with X, and take its r-jet at (p, p), which is the intersection of $\{p\} \times X''$ with the diagonal. Moreover, it is an $C^\infty(X')$-algebra isomorphism, if we endow $C^\infty(X_\Delta^{(r)})$ with the $C^\infty(X')$-module structure via the map $pr_1^* : C^\infty(X') \to C^\infty(X' \times X'')$.

Unfortunately the inverse of τ_1 cannot be written down in a clean way. For the construction of the inverse we need partition of unity. Take a locally finite over of X by local charts, so that one can write a section s of J^r as a sum of sections s_i, each supported in some local chart (U_i, x_i) of the cover. Then on U_i, s_i is of the form

$$s_i(p) = [a_0(p) + a_1(p)(x_i - p) + \cdots + a_r(p)(x_i - p)^r], \quad \forall\, p \in U_i.$$

We have correspondingly a function

$$f_i(x_i', x_i'') = a_0(x_i') + a_1(x_i')(x_i'' - x_i') + \cdots + a_r(x_i')(x_i'' - x_i')^r$$

on $U_i' \times U_i'' \subset X' \times X''$. The open subsets $U_i' \times U_i''$ also form a locally finite cover of an open neighborhood of the diagonal, so we can form $f = \sum_i f_i$, which is a function defined near the diagonal and thus gives an element in $C^\infty(X_\Delta^{(r)})$. Easy to check it is equal to $\tau^{-1}(s)$.

The only unsatisfying thing is that, in our definition of $C^\infty(X_\Delta^{(r)})$ or $I_\Delta^{(r)}$, knowledge about the special splitting of the ambient manifold is assumed. But we observe that the definition doesn't change if we take vector fields of arbitrary direction in (1)! (Exercise) So we come up with a more intrinsic definition

(3)
$$I_\Delta^{(r)} := \left\{ f \in C^\infty(X' \times X'') \;\middle|\; \begin{array}{c} V_1 V_2 \cdots V_l f|_\Delta = 0,\ \forall\, V_j \in C^\infty(T(X' \times X'')), \\ \forall\, 1 \le j \le l,\ 0 \le l \le r. \end{array} \right\}.$$

In this way one can immediately generalize the definition to the case of general embeddings $X \hookrightarrow Y$. Of course, the reader might already notice that the ideals we're talking about here are nothing but again powers of the ideal of functions vanishing along the submanifold. We don't want to emphasize this, however, since it's no longer true in the following discussion about holomorphic formal neighborhoods.

2. HOLOMORPHIC JET BUNDLES

Now let's consider holomorphic jet bundles. Let X be a complex manifold with structure sheaf $\mathcal{O}_X$ of germs of holomorphic functions. Mimicking what we did in the C^∞-case, for a given point $p \in X$, there is the space of r-jets of holomorphic functions at p,

$$\mathcal{J}_p^r := \mathcal{O}_p / \mathfrak{m}_p^{r+1},$$

where $\mathcal{O}_p$ is the stalk of $\mathcal{O}_X$ at p and $\mathfrak{m}_p \subset \mathcal{O}_p$ is the maximal ideal of germs of holomorphic functions vanishing at p. Under a local (holomorphic) chart (U, z_i) containing p, we have an isomorphism

$$\mathcal{J}_p^r \cong \mathbb{C}[z_1, \ldots, z_n]/(z_1 - p_1, \ldots, z_n - p_n)^{r+1}.$$

Let me be lazy again and assume $\dim_{\mathbb{C}} X = 1$. $\mathcal{J}^r$ as a smooth vector bundle has smooth sections which are locally of form

$$s(p) = [a_0(p) + a_1(p)(z-p) + \cdots + a_r(p)(z-p)^r] \in \mathcal{J}^r_p, \quad p \in U$$

for some local chart (U, z), where $a_0, \dots, a_r \in C^\infty(U)$. To give $\mathcal{J}^r_p$ its holomorphic structure, I just declare that s is holomorphic if and only if those a_i's are holomorphic for some (any) chart. It's an easy exercise to check the definition works. So locally the sections

$$[1], [z-p], \dots, [(z-p)^r]$$

give a local holomorphic frame of $\mathcal{J}^r$. Thus the $\overline{\partial}$-derivation on the Dolbeault complex $\Omega^{0,\bullet}(\mathcal{J}^r)$ can be written locally as

$$\overline{\partial} s = \overline{\partial} a_0 \otimes [1] + \overline{\partial} a_1 \otimes [z-p] + \cdots + \overline{\partial} a_r \otimes [(z-p)^r].$$

Can we fit $\mathcal{J}^r$ again into the diagonal picture? A first try is to consider all functions on $X' \times X''$ which are holomorphic along X''-direction but only smooth in X'-direction. Well, this works but again we appeal to the special feature of the product $X' \times X''$. To overcome this, we consider all smooth functions on $X' \times X''$, but then take the quotient by an appropriate equivalence relation so that we only memorize holomorphic derivatives of our functions. To begin with, one notice that there is an alternative definition of the fiber $\mathcal{J}^r_p$:

$$\mathcal{J}^r_p = C^\infty(X)/\mathfrak{a}^{(r)}_p$$

where $\mathfrak{a}^{(r)}_p$ is the closed ideal consisting of all smooth functions whose holomorphic derivatives vanish at p up to order r.

Next, one just modify (1) to define

$$\mathfrak{a}_r := \left\{ f \in C^\infty(X' \times X'') \;\middle|\; \begin{array}{c} \Delta^*(V_1 V_2 \cdots V_l f) = 0, \ \forall\, V_j \in C^\infty(T^{1,0}X''), \\ \forall\, 1 \le j \le l,\ 0 \le l \le r. \end{array} \right\}. \tag{4}$$

In other words, V_i's in the definition are $(1,0)$-tangent vector fields in the direction of X''. As before we can drop the restriction on directions as in (3), so we finally come up with

$$\mathfrak{a}_r := \left\{ f \in C^\infty(X' \times X'') \;\middle|\; \begin{array}{c} \Delta^*(V_1 V_2 \cdots V_l f) = 0, \ \forall\, V_j \in C^\infty(T^{1,0}(X' \times X'')), \\ \forall\, 1 \le j \le l,\ 0 \le l \le r. \end{array} \right\}. \tag{5}$$

Then we have a new model for $\mathcal{J}^r$ (more precisely, $C^\infty(X, \mathcal{J}^r)$):

$$\mathcal{A}(X^r_\Delta) := C^\infty(X' \times X'')/\mathfrak{a}_r.$$

Note that $\mathfrak{a}_r$ is just the ideal of smooth functions vanishing along the diagonal, but $\mathfrak{a}_r \neq \mathfrak{a}_0^{r+1}$.

As in the C^∞-case, we have a $C^\infty(X')$-algebra isomorphism

$$\sigma_1 : \mathcal{A}(X_\Delta^{(r)}) \xrightarrow{\cong} C^\infty(X, \mathcal{J}^r), \quad [f] \mapsto s_{[f]}, \tag{6}$$

such that

$$s_{[f]}(p) = j_p^r(f|_{\{p\}\times X''}),$$

which is however independent of the choice of representative function f. The inverse σ_1^{-1} can be constructed using partition of unity in the same as τ_1^{-1} in the previous section.

Next, I'll extend the algebra $\mathcal{A}^0(X_\Delta^{(r)}) = \mathcal{A}(X_\Delta^{(r)})$ to a differential graded algebra $\mathcal{A}^\bullet(X_\Delta^{(r)})$, which is isomorphic to the Dolbeault complex $\Omega^{0,\bullet}(\mathcal{J}^r)$, yet in such a way that it works for arbitrary embedding. Let me just throw the definition and explain in a moment. We define a closed dg-ideal $\mathfrak{a}_r^\bullet$ of the Dolbeault dga $\mathcal{A}^{0,\bullet}(X\times X)$ with $\mathfrak{a}_r^0 = \mathfrak{a}_r$ as the zero-th component:

$$\mathfrak{a}_r^k := \left\{ \omega \in \mathcal{A}^{0,k}(X\times X) \;\middle|\; \begin{array}{c} \Delta^*(L_{V_1}L_{V_2}\cdots L_{V_l}\omega) = 0,\ \forall\, V_j \in C^\infty(T^{1,0}(X\times X)), \\ \forall\, 1\le j\le l,\ 0\le l\le r. \end{array} \right\}$$

where Δ^* is the pullback map of differential forms. If we can check that $\mathfrak{a}_r^\bullet$ is invariant under the $\overline{\partial}$-derivation of $\mathcal{A}^{0,\bullet}(X\times X)$, then the quotient algebra

$$\mathcal{A}^\bullet(X_\Delta^{(r)}) := \mathcal{A}^{0,\bullet}(X\times X)/\mathfrak{a}_r^\bullet$$

is also a dga. Before we do that, let me point out that all this gadget works for *arbitrary* closed embedding $i : X \hookrightarrow Y$, once we substitute the ambient manifold $X\times X$ by Y and Δ^* by i^* in the definition of $\mathfrak{a}_r^\bullet$:

(7)

$$\mathfrak{a}_r^k = \mathfrak{a}_{X/Y,r}^k := \left\{ \omega \in \mathcal{A}^{0,k}(Y) \;\middle|\; \begin{array}{c} i^*(L_{V_1}L_{V_2}\cdots L_{V_l}\omega) = 0,\ \forall\, V_j \in C^\infty(T^{1,0}Y), \\ \forall\, 1\le j\le l,\ 0\le l\le r. \end{array} \right\}$$

Thus for any closed embedding $i : X \hookrightarrow Y$ and $r \in \mathbb{N}$ we can associate a "Dolbeault dga"

$$\mathcal{A}^\bullet(X_Y^{(r)}) := \mathcal{A}^\bullet(Y)/\mathfrak{a}_r^\bullet$$

which can be thought of as the Dolbeault complex on the rth-order formal neighborhood of X in Y.

Let's verify that $\mathfrak{a}_r^\bullet$ is indeed a dg-ideal. First notice that, if V is a $(1,0)$-vector field and ω is a $(0,k)$-form, then the Lie derivative $L_V\omega$ is still a $(0,k)$-form on Y. Moreover, this operation is linear with respect to V, i.e., if g is a smooth function, then

$$L_{gV}\omega = g\cdot L_V\omega.$$

Indeed, by Cartan's formula,

$$L_V\omega = \iota_V d\omega + d\iota_V\omega.$$

But $\iota_V\omega = 0$ since we're contracting a $(1,0)$-vector field with a $(0,k)$-form! So we have

$$L_V\omega = \iota_V d\omega = \iota_V(\partial\omega + \overline{\partial}\omega) = \iota_V\partial\omega,$$

which is obviously linear in V. We also use this equality to compute the commutator of $\overline{\partial}$ with L_V:

$$\begin{aligned} \overline{\partial}(L_V\omega) - L_V(\overline{\partial}\omega) &= \overline{\partial}(\iota_V\partial\omega) - \iota_V\partial\overline{\partial}\omega \\ &= \overline{\partial}(\iota_V\partial\omega) + \iota_V\overline{\partial}\partial\omega \\ &= (\overline{\partial}\circ\iota_V + \iota_V\circ\overline{\partial})\partial\omega \\ &= \iota_{\overline{\partial}V}\partial\omega \\ &= L_{\overline{\partial}V}\omega \end{aligned} \tag{8}$$

Let me explain the last two equalities. Here we extend the contraction ι to an operation

$$\iota_{(\cdot)}(\cdot) : \mathcal{A}^{0,k}(T^{1,0}Y) \times \mathcal{A}_Y^{1,l} \to \mathcal{A}_Y^{0,k+l}$$

such that

$$\iota_{\eta\otimes V}\xi = \eta \wedge (\iota_V\xi), \quad \forall\, \eta \in \mathcal{A}_Y^{0,0},\ \xi \in \mathcal{A}_Y^{1,\bullet},\ \forall\, V \in C^\infty(T^{1,0}Y).$$

Similarly, we can also define an extension of the Lie derivative

$$L_{(\cdot)}(\cdot) : \mathcal{A}^{0,k}(T^{1,0}Y) \times \mathcal{A}_Y^{0,l} \to \mathcal{A}_Y^{0,k+l}$$

by

$$L_{\eta\otimes V}\zeta := \iota_{\eta\otimes V}\partial\zeta = \eta \wedge L_V\zeta, \quad \forall\, \eta,\ \zeta \in \mathcal{A}_Y^{0,\bullet},\ \forall\, V \in C^\infty(T^{1,0}Y).$$

By these notations, one can show that

$$\iota_{\overline{\partial}V} = \overline{\partial}\circ\iota_V + \iota_V\circ\overline{\partial} = [\overline{\partial}, \iota_V] \tag{9}$$

(e.g., using local coordinate system). Note that ι_V is an operator on $\mathcal{A}_Y^{\bullet,\bullet}$ of degree -1, thus according the Koszul sign convention, the commutator of $\overline{\partial}$ and ι_V is indeed $\overline{\partial}\circ\iota_V + \iota_V\circ\overline{\partial}$ instead of the one with minus sign.

Remark 2.1. There is nothing fantastic about our extension of the Cartan's formula. In fact, one can identify the complex $\mathcal{A}_Y^{1,\bullet}$ with $\mathcal{A}_Y^{0,\bullet}(T^{1,0*}Y)$ via

$$\gamma : \mathcal{A}_Y^{0,\bullet}(T^{1,0*}Y) \xrightarrow{\cong} \mathcal{A}_Y^{1,\bullet}, \quad \omega\otimes\mu \mapsto \omega\wedge\mu$$

where $\omega \in \mathcal{A}_Y^{0,\bullet}$ and $\mu \in C^\infty(T^{1,0*}Y)$. There is a natural pairing $\mathcal{A}_Y^{0,\bullet}(T^{1,0}Y)$ with $\mathcal{A}_Y^{0,\bullet}(T^{1,0*}Y)$, denoted by

$$\langle\cdot,\cdot\rangle : \mathcal{A}_Y^{0,k}(T^{1,0}Y) \times \mathcal{A}_Y^{0,l}(T^{1,0*}Y) \to \mathcal{A}_Y^{0,k+l}$$

Then under the isomorphism γ, we can relate $\langle\cdot,\cdot\rangle$ with the contraction ι via

$$\langle\omega\otimes V, \eta\otimes\mu\rangle = (-1)^{|\eta|}\iota_{\omega\otimes V}(\eta\wedge\mu).$$

One can use this relation to translate the Lebniz rule of the Dolbeault differential $\overline{\partial}$ on $\langle\cdot,\cdot\rangle$ into the equality (9) about ι on $\mathcal{A}_Y^{1,\bullet}$.

What we've just done can be packaged into one single natural homomorphism of dg-Lie algebras

$$\theta : \mathcal{A}^{0,\bullet}(T^{1,0}Y) \to \mathcal{D}er_{\mathbb{C}}^{\bullet}(\mathcal{A}_Y^{0,\bullet}, \mathcal{A}_Y^{0,\bullet}).$$

In local holomorphic coordinates $z_1, \ldots, z_n$, it is given by

$$\theta\left(\phi\frac{\partial}{\partial z_i}\right)(f d\overline{z}_I) = L_{\phi\frac{\partial}{\partial z_i}}(f d\overline{z}_I) = \phi \wedge \frac{\partial}{\partial z_i} d\overline{z}_I.$$

The Dolbeault differential in $\mathcal{A}_Y^{0,\bullet}(T^{1,0}Y)$ corresponds, via the homomorphism θ, to the adjoint operator

$$[\overline{\partial}, -] : \mathcal{D}er_{\mathbb{C}}^{\bullet}(\mathcal{A}_Y^{0,\bullet}, \mathcal{A}_Y^{0,\bullet}) \to \mathcal{D}er_{\mathbb{C}}^{\bullet+1}(\mathcal{A}_Y^{0,\bullet}, \mathcal{A}_Y^{0,\bullet})$$

Observe that if we substitute, in the definition (7) of $\mathfrak{a}_r^{\bullet}$, those L_{V_i}'s by Lie derivatives $L_{\mathcal{V}_i}$ with respect to $\mathcal{V}_i \in \mathcal{A}_Y^{0,\bullet}(T^{1,0}Y)$, nothing will be changed. Indeed, if $\omega \in \mathfrak{a}_r^{\bullet}$ and $\mathcal{V}_i \in \mathcal{A}_Y^{0,\bullet}(T^{1,0}Y)$, $1 \leq i \leq l$, $l \leq r$, then we also have

$$i^* L_{\mathcal{V}_1} \cdots L_{\mathcal{V}_l} \omega = 0.$$

Moreover, by (8), the commutator

$$[\overline{\partial}, L_{V_1} \cdots L_{V_l}] = \sum_{i=1}^{l} L_{V_1} \cdots L_{\overline{\partial} V_i} \cdots L_{V_l}$$

is still a differential operator on $\mathcal{A}_Y^{0,\bullet}$ of order $\leq l$. Thus if $\omega \in \mathfrak{a}_r^{\bullet}$, then

$$i^* L_{V_1} \cdots L_{V_l} \overline{\partial}\omega = \overline{\partial} i^* L_{V_1} \cdots L_{V_l} \omega - \sum_{i=1}^{l} i^* L_{V_1} \cdots L_{\overline{\partial} V_i} \cdots L_{V_l} \omega = 0,$$

for any $0 \leq l \leq r$, which means that $\overline{\partial}\omega$ also lies in $\mathfrak{a}_r^{\bullet}$. Hence our Dolbeault dga is well-defined.

Let's get a taste of this abstractly defined dga $\mathcal{A}^{\bullet}(X_Y^{(r)})$ by studying an easy example. Let $X = \mathbb{C}$ and $Y = \mathbb{C} \times \mathbb{C} = \{(z, w) \mid z, w \in \mathbb{C}\}$ and the embedding

$$i : X \to Y, \quad i(z) = (z, 0)$$

identifies X with the submanifold of Y defined by the equation $w = 0$. A smooth function f on Y belongs to $\mathfrak{a}_r^0$ if and only if, by means of Taylor expansion, it can be written as

$$f(z, w) = w^{r+1} \cdot g(z, w) + \overline{w} \cdot h(z, w)$$

for some $g,\ h \in C^\infty(Y)$. Hence $\mathfrak{a}_r^0$ is the ideal of $\mathcal{A}_Y^{0,0} = C^\infty(Y)$ generated by functions w^{r+1} and $\overline{w}$ and there is an isomorphism

$$\mathcal{A}^0(X_Y^{(r)}) = \mathcal{A}_Y^{0,0}/\mathfrak{a}_r^0 \xrightarrow{\cong} \mathcal{A}_X^{0,0} \otimes_{\mathbb{C}} \mathbb{C}[w]/(w)^{r+1}, \quad [f]_r \mapsto \sum_{k=0}^{r} \frac{1}{k!}\frac{\partial^k f}{\partial w^k}(z,0)\cdot w^k.$$

For a smooth $(0,1)$-form

$$\omega = f(z,w)d\overline{z} + g(z,w)d\overline{w},$$

it lies in $\mathfrak{a}_r^1$ if and only if $f \in \mathfrak{a}_r^0$. No condition to put on g since $i^* d\overline{w} = 0$. Thus

$$\mathfrak{a}_r^1 = \mathfrak{a}_r^0 \cdot d\overline{z} + \mathcal{A}_Y^{0,0}\cdot d\overline{w}$$

and there is an isomorphism

$$\mathcal{A}^1(X^{(r)})_Y = \mathcal{A}_Y^{0,1}/\mathfrak{a}_r^1 \xrightarrow{\cong} \mathcal{A}_X^{0,1} \otimes_{\mathbb{C}} \mathbb{C}[w]/(w)^{r+1}$$

defined by

$$[f d\overline{z} + g d\overline{w}]_r \mapsto \sum_{k=0}^{r} \frac{1}{k!}\frac{\partial^k f}{\partial w^k}(z,0)d\overline{z}\otimes w^k.$$

Put these two isomorphisms together, we obtain an isomorphism between dgas

$$\mathcal{A}^\bullet(X_Y^{(r)}) = \mathcal{A}_X^{0,\bullet} \otimes_{\mathbb{C}} \mathbb{C}[w]/(w)^{r+1}.$$

Now let's go back to the case of the diagonal embedding and see how to identify our new-defined dga $\mathcal{A}^\bullet(X_\Delta^{(r)})$ with the Dolbeault complex of $\mathcal{J}^r$. It's of the same spirit as definition (6) of τ_1 in the previous section, but one needs a little bit more than that to deal with $(0,k)$-forms with $k \geq 1$. Let me illustrate the case when $k = 1$. General cases follow easily from a similar argument. Given $[\omega] \in \mathfrak{a}_r^1$, the goal is to construct a corresponding a section s_ω of the vector bundle $\mathrm{Hom}(T^{0,1}X, \mathcal{J}^r)$. For any $W \in T_p^{0,1}X$ at some point $p \in X$, we form $\Delta_* W \in T_{(p,p)}^{0,1}(X' \times X'')$, a $(0,1)$-tangent vector along the diagonal at $(p,p) \in X' \times X''$ by pushforward. We can always extend this vector to a local 'anti-holomorphic' $(0,1)$-vector field $\widetilde{W}$ on an open neighborhood of (p,p), such that for any $(1,0)$-vector field V on $X' \times X''$, the Lie bracket $[V, \widetilde{W}]$ is again of $(1,0)$-type (Exercise!). Finally contract $\widetilde{W}$ with ω, restrict the resulted function along $\{p\} \times X''$ and take the jet at (p,p). In short, we have

$$s_{[\omega]}(W) = j_p^r((\iota_{\widetilde{W}}\omega)|_{\{p\}\times X''}).$$

One can check this definition is independent of the choice of $\widetilde{W}$ and representative form ω. To see this, one only needs to notice that, because of the way we choose $\widetilde{W}$, there is

$$V_1 V_2 \cdots V_l(\iota_{\widetilde{W}}\omega) = \iota_{\widetilde{W}}(L_{V_1} L_{V_2} \cdots L_{V_l}\omega)$$

for any $V_i \in C^\infty(T^{1,0}(X \times X))$. The case of $\mathcal{A}^k$ for arbitrary k is similar. Thus we obtain a homomorphism of graded algebra

$$\sigma_1 : \mathcal{A}^\bullet(X_\Delta^{(r)}) \to \Omega^{0,\bullet}(\mathcal{J}^r), \quad [\omega] \mapsto s_{[\omega]}$$

More explicitly, under some local chart (U, z) containing p, hence $(U' \times U'', z', z'')$ containing (p, p), suppose

$$\omega = f(z', z'')d\bar{z}' + g(z', z'')d\bar{z}'' \in \Omega^{0,1}(U' \times U''),$$

which defines an equivalence class $[\omega] \in \mathcal{A}^1(U_\Delta^{(r)})$, then

$$s_{[\omega]}(p) = d\bar{z} \otimes \left[\sum_{i=0}^{r} \frac{1}{i!}\left(\frac{\partial^i f}{\partial z''^i}(p,p) + \frac{\partial^i g}{\partial z''^i}(p,p)\right)(z-p)^i\right], \quad \forall\, p \in U.$$

Using this local expression, one can check that σ_1 is also a homomorphism of dgas. Also from this we see that there is an isomorphism of dgas

$$\mathcal{A}^\bullet(U_\Delta^{(r)}) \cong \Omega^{0,\bullet} \otimes \mathbb{C}[dz]/(dz)^{r+1},$$

or written in more general form for arbitrary dimension,

$$\mathcal{A}^\bullet(U_\Delta^{(r)}) \cong \Omega_U^{0,\bullet}(\oplus_{i=0}^r S^i T^*X), \tag{10}$$

where we identify T^*X with the $(1,0)$-cotangent bundle $\Omega^{1,0}X$. Moreover, $\mathcal{A}^\bullet(X_\Delta^r)$ is an $\mathcal{A}^\bullet(X')$-algebra via $pr'^* : \mathcal{A}^\bullet(X') \to \mathcal{A}^\bullet(X_\Delta^r)$ and σ_1 respects this structure. Hence the inverse of σ_1 can be constructed by extending σ_1^{-1} on the zero-th component in the following way:

$$\sigma_1^{-1}(\eta \otimes s) = [pr'^*(\eta)] \cdot \sigma_1^{-1}(s), \quad \forall\, \eta \in \Omega^{0,\bullet}(X'),\ s \in C^\infty(\mathcal{J}^r).$$

3. Kapranov's Theorem

The rest of the notes will contribute to the understanding the holomorphic structure of the formal neighborhood of the diagonal embedding, i.e., the dga $(\mathcal{A}^\bullet(X_\Delta^{(\infty)}), \bar{\partial})$. As we've seen before, at least in some local chart, we have some isomorphism

$$\mathcal{A}^\bullet(U_\Delta^{(\infty)}) \cong \Omega_U^{0,\bullet}(\hat{S}(T^*X))$$

where

$$\hat{S}(T^*X) = \prod_{i=0}^{\infty} S^i T^* X$$

is the bundle of complete symmetric algebras generated by T^*X and the differential on $\Omega_U^{0,\bullet}(\hat{S}(T^*X))$ is the usual $\overline{\partial}$-derivation induced from that of T^*X. This isomorphism is just obtained by taking the inverse limit of isomorphisms (10) for all formal neighborhoods of finite orders.

In the global case, however, we don't have such an isomorphism if X is not affine. By this, I mean that one can always find an isomorphism between the two as *graded algebras* (actually there are plenty of such isomorphisms), but it might not be compatible with the usual $\overline{\partial}$ on $\hat{S}(T^*X)$. What we can do is to correct the holomorphic structure on $\hat{S}(T^*X)$ to make it compatible. It may sound trivial because one can just transfer the differential on $\mathcal{A}^\bullet(U_\Delta^{(\infty)})$ to one on $\Omega_U^{0,\bullet}(\hat{S}(T^*X))$ via the chosen isomorphism. It requires some work, however, to write down explicitly the transferred differential. For this purpose we need to pick an isomorphism with transparent geometric meaning so that the new differential could be expressed in terms of the geometry of X.

Now suppose that X is equipped with a Kähler metric h. Let ∇ be the canonical $(1,0)$-connection in TX associated with h, so that

$$[\nabla, \nabla] = 0 \text{ in } \Omega^{2,0}(\mathrm{End}(TX)). \tag{11}$$

and it is torsion-free, which is equivalent to the condition for h to be Kähler.

For most of the time here, however, I will use ∇ as a connection on the cotangent bundle T^*X. In other words, I think of ∇ as a differential operator

$$\nabla : T^*X \to T^*X \otimes T^*X$$

such that

$$\langle \nabla(\alpha), V \otimes W \rangle = \langle \nabla_V \alpha, W \rangle$$

where α is any section of T^*X and V and W are any sections of TX. I can also apply ∇ iteratively to get sections of higher order tensors of T^*X.

Moreover, I will put a constant family of ∇'s on $X' \times X''$ along the X''-fibers, so that we get a differential operator only in X''-directions with respect to the decomposition $T(X' \times X'') = TX' \oplus TX''$:

$$\nabla : T^*X'' \to T^*X'' \otimes T^*X''$$

where I still use the same notation ∇. Then we can define

$$\exp^* : \mathcal{A}^\bullet(X_{X\times X}^{(\infty)}) \xrightarrow{\cong} \Omega^{0,\bullet}(\hat{S}(T^*X))$$

by

$$\exp_\sigma^*([f]_\infty) = (\Delta^* f, \Delta^*\nabla f, \Delta^*(\nabla)^2 f, \cdots, \Delta^*(\nabla)^n f, \cdots) \in \hat{S}(T^*X)$$

where

$$\nabla f := \partial'' f$$

is the $(1,0)$-differential of f along X''-fibers, so it lies in T^*X''. And

$$\nabla^i f = \nabla^{i-1}\partial'' f, \quad i \geq 2$$

To show that $\nabla^i f$ indeed lies in symmetric tensors, one has to resort to the torsionfreeness and flatness of ∇.

The curvature of $\tilde{\nabla} = \nabla + \overline{\partial}$ is just

$$R = [\overline{\partial}, \nabla] \in \Omega^{1,1}(\mathrm{End}(TX)) = \Omega^{0,1}(\mathrm{Hom}(TX \otimes TX, TX))$$

which is a Dolbeault representative of the Atiyah class α_{TX} of the tangent bundle. In particular one has the Bianchi identity:

$$\overline{\partial} R = 0 \text{ in } \Omega^{0,2}(\mathrm{Hom}(TX \otimes TX, TX))$$

Actually, by the torsion-freeness we have

$$R \in \Omega^{0,1}(\mathrm{Hom}(S^2TX, TX))$$

We then define tensor fields R_n, $n \geq 2$, as higher covariant derivatives of the curvature:

$$R_n \in \Omega^{0,1}(\mathrm{Hom}(S^2TX \otimes TX^{\otimes(n-2)}, TX)), \quad R_2 := R, \quad R_{i+1} = \nabla R_i \tag{12}$$

In fact R_n is totally symmetric, i.e.,

$$R_n \in \Omega^{0,1}(\mathrm{Hom}(S^nTX, TX)) = \Omega^{0,1}(\mathrm{Hom}(T^*X, S^nT^*X))$$

by the flatness of ∇ (**??**).

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