NOTES ON DEFORMATION QUANTIZATION

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ABSTRACT. ...

CONTENTS

1. DEFORMATION THEORY

1.1. **Deformation of associative algebras.** Fix a field k of characteristic zero. Let A be an associative algebra. A *formal deformation* of A is an associative $k[\![\hbar]\!]$-algebra structure on $A[\![\hbar]\!]$ such that $A \simeq A[\![\hbar]\!]/(\hbar)$ as algebras. We denote the multiplication on $A[\![\hbar]\!]$ by the star product $\star : A[\![\hbar]\!] \times A[\![\hbar]\!] \to A[\![\hbar]\!]$, then it is determined by its values on the subspace $A \subset A[\![\hbar]\!]$. We write for any $f, g \in A$,

$$f \star g = fg + B_1(f,g)\hbar + B_2(f,g)\hbar^2 + \cdots + B_n(f,g)\hbar^n + \cdots . \tag{1}$$

where $B_n : A \otimes A \to A$.

Let J be the group of $k[\![\hbar]\!]$-module automorphisms E of $A[\![\hbar]\!]$, such that

$$E(f) \equiv f \mod tA[\![\hbar]\!],$$

i.e., E is of the form

$$E(f) = f + E_1(f)\hbar + E_2(f)\hbar^2 + \cdots , \quad E_i \in \mathrm{Hom}_k(A, A).$$

Key words and phrases. ...

Then J acts on the set of all formal deformations. Any $E \in J$ transforms a star product $\star$ to $\star'$ via

$$f \star' g = E(E^{-1}(f) \star E^{-1}(g)). \tag{2}$$

In this case, we say that *star* and $\star'$ are gauge equivalent. We want to classify formal deformations of A up to gauge equivalences.

Let us first look at the *first-order deformations* of A, i.e., $k[\![\hbar]\!]$-algebra structures on $A[\![\hbar]\!]/(\hbar) = A \oplus \hbar \cdot A$. If we expand the associativity condition $(f \star g) \star h = f \star (g \star h)$ according to (1) and compare the coefficient of the first-order term, we get

$$B_1(fg, h) + B_1(f, g)h = fB_1(g, h) + B_1(f, gh), \tag{3}$$

or equivalently,

$$fB_1(g, h) - B_1(fg, h) + B_1(f, gh) - B_1(f, g)h = 0. \tag{4}$$

This means that $B_1 : A \otimes A \to A$ is a *Hochschild 2-cocycle*. Now suppose there is a gauge transform $E = \mathrm{Id}_A + \hbar E_1$ on B_1, which maps a star product $\star$ with first coefficient B_1 to $\star'$ with first coefficient B_1'. Then we have

$$B_1'(f, g) = B_1(f, g) - fE_1(g) + E_1(fg) - E_1(f)g. \tag{5}$$

Hence B_1' differs from B_1 with a Hochschild 2-coboundary.

Theorem 1.1. *The set of gauge equivalence classes of first-order deformation of an associative algebra A is bijective to the second Hochschild cohomology $HH^2(A, A)$.*

1.2. **Hochschild complex and dg-Lie algebra.** Let A be as before. The *Hochschild cochain complex* $C^\bullet(A, A)$ is defined to be

$$C^n(A, A) := \mathrm{Hom}_k(A^{\otimes n}, A)$$

with the differential $b : C^n(A, A) \to C^{n+1}(A, A)$ given by

$$b\phi(a_1, a_2, \cdots, a_{n+1}) = a_1\phi(a_2, \cdots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \phi(a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) + (-1)^{n+1}\phi(a_1, \cdots, a_n)a_{n+1},$$

for any $\phi \in C^n(A, A)$. The cohomology of $(C^\bullet(A, A), b)$ is the Hochschild cohomology $HH^\bullet(A, A)$ of the algebra A.

Now suppose we have a formal deformation $(A[\![\hbar]\!], \star)$ of A. Denote by $\mu : A \otimes A \to A$ the original multiplication of A and by $\mu_\hbar : A[\![\hbar]\!] \otimes_{k[\![\hbar]\!]} A[\![\hbar]\!] \to A[\![\hbar]\!]$ the star product. We will rewrite the associativity condition of $\mu_\hbar$ in terms of a DGLA structure on the Hochschild complex $(C^\bullet(A,A),b)$. Let ϕ be a Hochschild p-cochain and φ be a Hochschild q-cochain. The Gerstenhaber product of ϕ and φ is the $(p+q-1)$-cochain defined by

$$\phi\circ\varphi(a_1,\cdots,a_{p+q-1})=\sum_{i=0}^{p-1}(-1)^{i(q+1)}\phi(a_1,\cdots,a_i,\varphi(a_{i+1},\cdots,a_{i+q}),a_{i+q+1},\cdots,a_{p+q-1}).$$

The *Gerstenhaber bracket* is then defined by

$$[\phi,\varphi]=\phi\circ\varphi-(-1)^{(p-1)(q-1)}\varphi\circ\phi$$

Definition 1.2. A *differential graded Lie algebra (DGLA)* is a complex $(L^\bullet,d)$ endowed with a Lie bracket

$$[\ ,\ ]:L^\bullet\otimes L^\bullet\to L^\bullet$$

of degree 0 which is

- antisymmetric: $[x,y]=(-1)^{|x||y|}[y,x]$,
- Jacobi identity:
$$[x,[y,z]]=[[x,y],z]+(-1)^{|x||y|}[y,[x,z]],$$
- compatibility:
$$d[x,y]=[dx,y]+(-1)^{|x|}[x,dy].$$

Lemma 1.3. *The shifted Hochschild complex $C^\bullet(A,A)[1]=C^{\bullet+1}(A,A)$ with the Gerstenhaber bracket is a DGLA.*

Note that the differential b can be rewritten in terms of the Gerstenhaber bracket and the multiplication μ:

$$d\phi=\pm[\mu,\phi]$$

We can form $C^\bullet(A,A)[\![\hbar]\!]$, the Hochschild complex depending formally on $\hbar$, and extend the differential b and the Gerstenhaber bracket into a $k[\![\hbar]\!]$-linear one on the new complex, so that it is again a DGLA. Regard $\mu_\hbar$ as an element in $C^2(A,A)[\![\hbar]\!]$, then we have

$$[\mu_\hbar,\mu_\hbar](f,g,h)=2(\mu_\hbar(\mu_\hbar(f,g),h)-\mu_\hbar(f,\mu_\hbar(g,h)))=0$$

since $\mu_\hbar$ is associative. Thus we have

$$[\mu_\hbar,\mu_\hbar]=0\in C^3(A,A)[\![\hbar]\!].\tag{6}$$

If we write

$$\mu_\hbar(f,g) = fg + B(f,g), \quad \text{or} \quad \mu_\hbar = \mu + B, B \in C^2(A,A)\hat{\otimes}\mathfrak{m},$$

where $\mathfrak{m} = \hbar k[\![\hbar]\!]$ is the maximal ideal of $k[\![\hbar]\!]$, then (6) can be rewritten as the *Maurer-Cartan equation*

$$bB + \frac{1}{2}[B,B] = 0, \tag{7}$$

and we say that $B \in C^1(A,A)[1]\hat{\otimes}\mathfrak{m}$ is a *Maurer-Cartan element* of the DGLA $L_A^\bullet = C^\bullet(A,A)[1]\hat{\otimes}\mathfrak{m}$ (which is a subDGLA of $C^\bullet(A,A)[1]\hat{\otimes}k[\![\hbar]\!]$). On the other hand, notice that the gauge action (2) on the set of $\mu_\hbar$'s is the exponential of the action of the pronilpotent Lie algebra $L_A^0 = C^1(A,A)\hat{\otimes}(\hbar)$ given by

$$\rho_X(\mu_\hbar) = [X, \mu + B] = [X,B] - bX \in L_A^1, \quad X \in L_A^0. \tag{8}$$

Thus we arrive at the following general definition.

Definition 1.4. Let $(L^\bullet, d)$ be a (nilpotent/pronilpotent) DGLA. The set of all Mauere-Cartan elements of $L^\bullet$ is defined to be

$$MC(L^\bullet) = \{x \in L^1 | dx + \frac{1}{2}[x,x] = 0\}.$$

The gauge action of $\exp L^0$ on $MC(L^\bullet)$ is the affine action on L^1 whose differential is the Lie algebra homomorphism

$$L^0 \to \mathrm{Lie}(\mathrm{Aff}(L^1)), \quad X \mapsto (x \mapsto [X,x] - dX).$$

The set of formal deformations governed by $L^\bullet$ over the formal disk $\mathrm{Spf}\, k[\![\hbar]\!]$ is

$$\mathrm{Def}_L(k[\![\hbar]\!]) := MC(L^\bullet\hat{\otimes}\mathfrak{m})/\exp(L^0\hat{\otimes}\mathfrak{m}).$$

Note that if $f : L_1^\bullet \to L_2^\bullet$ is a homomorphism of DGLAs, then it induces map $\mathrm{Def}_{L_1} \to \mathrm{Def}_{L_2}$.

1.3. **Deformation quantization of Poisson manifolds.** Our primary interest is in the deformation of the algebra of smooth functions $A = C^\infty(M)$ (which is commutative!) on a smooth manifold with extra structures. For this purpose, the usual Hochschild complex $C^\bullet(A,A)$ is too big. Instead, we use a subcomplex $D_{poly}^\bullet$ of the shifted Hochschild complex $C^\bullet(A,A)[1]$, which consists of those cochains that are polydifferential operators. For instance, a cochain in D^n_{poly} under some local coordinates (x_i) is of the form

$$f_0 \otimes \cdots \otimes f_n \mapsto \sum_{I_0,\cdots,I_n} C^{I_0,\cdots,I_n}(x) \cdot \partial_{I_0}(f_0) \cdots \partial_{I_n}(f_n).$$

The corresponding cohomology groups will be referred as the Hochschild cohomology $HH^\bullet(A, A)$. This means that in the expansion of the star product (1), we require B_n to be polydifferential operators. Suppose there is a formal deformation of $C^\infty(M)$ with a star product $\star$, we define a *Poisson bracket* on $C^\infty(M)$ by

$$\{f, g\} = \frac{f \star g - g \star f}{2\hbar}\bigg|_{\hbar=0} = \frac{1}{2}(B_1(f, g) - B_1(g, f)) = B_1^-(f, g), \quad \forall\ f, g \in C^\infty(M).$$

where $B_1^-(f, g)$ is the antisymmetric part of B_1. Notice that the gauge action (5) only changes the symmetric part of B_1. In fact, the symmetric part of B_1 can always be eliminated by a gauge transform. So from now on let us assume B_1 is antisymmetric. The Poisson bracket acts as derivations in both parameters, i.e.,

$$\{f, gh\} = \{f, g\}h + g\{f, h\},$$

and it satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

To see this, we take the commutators of the star product

$$\begin{aligned}
[f, g] &= f \star g - g \star f \\
&= \hbar(B_1(f, g) - B_1(g, f)) + \hbar^2[B_2(f, g) - B_2(g, f)] + O(\hbar^3) \\
&= \hbar 2B_1(f, g) + \hbar^2[B_2(f, g) - B_2(g, f)] + O(\hbar^3),
\end{aligned}$$

so

$$[f, g] \star h = \hbar 2B_1(f, g)h + \hbar^2((B_2(f, g) - B_2(g, f))h + 2B_1(B_1(f, g), h)) + O(\hbar^3),$$

and

$$h \star [f, g] = \hbar 2hB_1(f, g) + \hbar^2(h(B_2(f, g) - B_2(g, f)) + 2B_1(h, B_1(f, g))) + O(\hbar^3).$$

Hence

$$[[f, g], h] = \hbar^2 4B_1(B_1(f, g), h) + O(\hbar^3) = \hbar^2 4\{\{f, g\}, h\} + O(\hbar^3),$$

and the Jacobi identity for the Poisson bracket comes from the $\hbar^2$ term of the Jacobi identity for the commutator of $\star$. We rephrase the Poisson bracket as a bivector field $\alpha \in \Gamma(\wedge^2 TM)$, such that $\langle \alpha, df \otimes dg\rangle = B_1^-(f,g) = \{f,g\}$. Then the Jacobi identity can be written as $[\alpha,\alpha]_{SN} = 0$. The bracket $[,]_{SN}$ here is the Schouten-Nijenhuis bracket which will be introduced in due course.

In other words, we have bijections between the set of Poisson structures , the second Hochschild cohomology $HH^2(A,A)$ and the gauge equivalence classes of all first-order deformations.

Questions 1.5. Given a Poisson structure on M, can it always be lifted to a formal deformation of $C^\infty(M)$?

Example 1.6 (**The Moyal-Weyl quantization**)**.** Let us start with the special case when the manifold is $V = \mathbb{R}^2 = \{(p,q)\}$ equipped with the symplectic form $\omega = 2dp \wedge dq$. The corresponding Poisson bivector is $\alpha = 2\partial_p \wedge \partial_q$. The poisson bracket on $C^\infty(V)$ is given by

$$\{f,g\} = \frac{\partial f}{\partial x_p}\frac{\partial g}{\partial x_q} - \frac{\partial f}{\partial x_q}\frac{\partial g}{\partial x_p}.$$

A relevant noncommutative algebra here is the *Weyl algebra*

$$A_\hbar = \mathbb{R}\langle p,q\rangle[\![\hbar]\!]/\langle pq - qp - 2\hbar\rangle.$$

We show that it is a deformation of the algebra $\mathbb{R}[V] = S(V^*)$ of polynomial functions over V. There is a $\mathbb{R}[\![\hbar]\!]$-linear symmetrization map

$$Sym : \mathbb{R}[V][\![\hbar]\!] \to A_\hbar$$

given by

$$Sym(v_1 v_2 \cdots v_n) = \frac{1}{n!}\sum_\sigma [v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}] \in A_\hbar.$$

By some filtration arguement one can see that this is a isomorphism of $\mathbb{R}[\![\hbar]\!]$-modules. Thus we can define a star product on $\mathbb{R}[V]$ by

$$f \star g = Sym^{-1}(Sym(f) \cdot Sym(g)).$$

On the other hand, as a subalgebra of $C^\infty(V)$, $\mathbb{R}[V]$ is invariant under the Poisson bracket. We show that the first-order term of the expansion of the star product on $\mathbb{R}[V]$ is equal to the Poisson bracket, i.e.,

$$\left.\frac{f\star g - g\star f}{2\hbar}\right|_{\hbar=0} = \{f,g\}, \quad \forall\ f,g \in \mathbb{R}[V]. \tag{9}$$

Notice that both sides of (9) are derivations with respect to f, g and the usual commutative product of functions. Thus we only need to check the special case when $f(p,q) = p$ and $g(p,q) = q$. Indeed, in $A_\hbar$ we have

$$Sym(p \star q - q \star q) = p \otimes q - q \otimes p = 2\hbar,$$

thus

$$p \star q - q \star p = 2\hbar = 2\hbar\{p, q\}.$$

One can show that the coefficients B_n in the expansion of the star product are differential operators in f and g, so we can use this expansion to extend the star product to $C^\infty(V)$, which is the Moyal-Weyl product. It is given by the formula

$$\begin{aligned} f \star g &= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left(\frac{\partial}{\partial p'} \frac{\partial}{\partial q''} - \frac{\partial}{\partial q'} \frac{\partial}{\partial p''} \right)^n f(p', q') g(p'', q'') \Big|_{\substack{p'=p''=p \\ q'=q''=q}} \\ &= \mu \circ e^{\hbar \alpha} \circ (f \otimes g), \end{aligned}$$

where μ is the usual commutative product of functions. To see it, let us consider the star product of $f(p,q) = e^{ap+bq}$ and $g(p,q) = e^{cp+dq}$, where a, b, c and d are arbitrary real numbers. We write $x = ap + bq$ and $y = cp + dq$. Then the commutator $[x, y] = 2\hbar(ad - bc)$ (with respect to $\star$) is a central element in $C^\infty(V)$ and all higher iterations of the commutator vanish. By Campbell-Baker-Hausdorff theorem, we have

$$e^x \star e^y = e^{x+y+\frac{1}{2}[x,y]} = e^{\frac{1}{2}[x,y]} e^{x+y} = e^{\hbar(ad-bc)} fg.$$

Hence

$$\begin{aligned} f \star g &= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} (ad - bc)^n fg \\ &= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left(\frac{\partial}{\partial p'} \frac{\partial}{\partial q''} - \frac{\partial}{\partial q'} \frac{\partial}{\partial p''} \right)^n f(p', q') g(p'', q'') \Big|_{\substack{p'=p''=p \\ q'=q''=q}}. \end{aligned}$$

More generally, let $M = \mathbb{R}^d$ and $\alpha = \sum_{i,j} \alpha^{i,j} \partial_i \wedge \partial_j$ is a Poisson bivector with constant coefficients, where $\alpha^{i,j} = -\alpha^{j,i}$. The corresponding Poisson bracket is given by

$$\{f, g\} = \langle \alpha, df \otimes dg \rangle = \sum_{i,j} \alpha^{i,j} \partial_i f \partial_j g.$$

The Moyal-Weyl product is given by the formula

$$\begin{aligned} f \star g &= fg + \hbar \sum_{i,j} \alpha^{i,j} \partial_i(f)\partial_j(g) + \frac{\hbar^2}{2}\alpha^{i,j}\alpha^{k,l}\partial_i\partial_k(f)\partial_j\partial_l(g) + \cdots \\ &= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{i_1,\cdots,i_n;j_1,\cdots,j_n} \prod_{k=1}^{n} \alpha^{i_k j_k} \left(\prod_{k=1}^{n} \partial_{i_k}\right)(f) \cdot \left(\prod_{k=1}^{n} \partial_{j_k}\right)(g) \\ &= \mu \circ e^{\hbar\alpha} \circ (f \otimes g). \end{aligned}$$

2. Kontsevich's formality theorem

We have seen that $H^1(D^\bullet_{poly}) = HH^2(A,A)$ is isomorphic to $\wedge^2 TM$. In fact, there is a more general result due to Hochschild-Kostant-Rosenberg, of which a variation used by Kontsevich is as follows. Let

$$T^n_{poly} = \Gamma(M, \wedge^{n+1}TM), \quad n \geq -1.$$

Then we can define a map of cochain complexes

$$I_{HKR} : (T^\bullet_{poly}, 0) \to (D^\bullet_{poly}, b)$$

by

$$I_{HKR}(\xi_0 \wedge \cdots \wedge \xi_n) = \left(f_0 \otimes \cdots \otimes f_n \mapsto \frac{1}{(n+1)!} \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{i=0}^{n} \xi_{\sigma(i)}(f_i) \right).$$

Theorem 2.1 (HKR). *Then I_{HKR} is a quasi-isomorphism between complexes.*

Since $D^\bullet_{poly}$ is a DGLA, its cohomology group $T^\bullet_{poly}$ admits a graded Lie algebra structure, which is given by the *Schouten-Nijenhuis bracket*:

$$[\xi_0\wedge\cdots\wedge\xi_k, \eta_0\wedge\cdots\wedge\eta_l]_{SN} = \sum_{i=0}^{k}\sum_{j=0}^{l}(-1)^{i+j+k}[\xi_i,\eta_j]\wedge\xi_0\wedge\cdots\wedge\hat{\xi}_i\wedge\cdots\wedge\xi_k\wedge\eta_0\wedge\cdots\wedge\hat{\eta}_j\wedge\cdots\wedge\eta_l.$$

for $k, l \geq 0$. For $k \geq 0$,

$$[\xi_0 \wedge \cdots \wedge \xi_k, h]_{SN} = \sum_{i=0}^{k} (-1)^i \xi_i(h) \cdot \xi_0 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_k.$$

It is tempting to say that I_{HKR} is a homomorphism of DGLAs, then we can map a Maurer-Cartan element of $T^\bullet_{poly}\hat{\otimes}\mathfrak{m}$, which is a Poisson bivector α, to a Maurer-Cartan element of $D^\bullet_{poly}\hat{\otimes}\mathfrak{m}$, which is a formal deformation of $C^\infty(M)$, and hence prove answer the Question 1.5! Unfortunately it is not the case... but Kontsevich showed that I_{HKR} can be fixed by an L_∞-*morphism*, which gives bijection between deformations arised from the two DGLAs.

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