# INTERACTIONS BETWEEN LIE THEORY AND ALGEBRAIC GEOMETRY

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## 1. Introduction

My primary research interests lie in the interactions of complex/algebraic geometry with Lie theory and representation theory in the spirit of noncommutative geometry, derived algebraic geometry and mathematical physics.

Both Lie theory and algebraic geometry have been at the center of the 20th-century mathematical studies. Entering the 21th century, the two subjects came across each other in a surprising way, due to the phenomenal work of Kontsevich on deformation quantization of Poisson manifolds ([45]) and the birth of derived algebraic geometry ([46]).

We start from the following observation: in algebraic geometry, the well-known Grothendieck-Riemann-Roch (GRR) theorem involves the Todd class, which is a characteristic class defined by a certain formal power series. The same type of class also appears in the Atiyah-Singer index theorem, which is a generalization of the Riemann-Roch theorem and one of the most important theorems of the 20th century. On the other hand, the exact same power series also shows up in Lie theory in a fundamental way, long before the GRR theorem was formulated. Namely, it appears in in the differential of the exponential map of a Lie group the Baker-Campbell-Hausdorff (BCH) formula in Lie theory. It (or its inverse) also plays an essential role in the Duflo-Kirillov isomorphism ([28]).

Due to the work of many people, including but not limited to Kapranov [40], Kontsevich [45], Markarian [48], and Ramadoss [53], it is now clear that the same power series appearing in the two seemly unrelated cases is not a coincidence. The phenomenon can be explained by a Lie algebra structure on the shifted tangent bundle $TX[-1]$ of any complex manifold discovered by Kapranov [40], which arises from the geometry of the diagonal embedding $X \hookrightarrow X \times X$. One of my projects is to generalize this to the case of an arbitrary closed embedding $X \hookrightarrow Y$ of complex manifolds, where more richer structures and phenomena emerge while the situation is more complicated at the same time.

The second subject is another intersection of Lie theory and algebraic geometry, but now the interaction is in the opposite direction. I propose a study of representations of semisimple Lie groups from a geometric perspective. It has been a long time since Mackey observed the surprising analogy of the representation theory of a real semisimple Lie group $G_{\mathbb{R}}$ and the representation theory of its Carton motion group. I have discovered that the mysterious seemingly coincidence can be explained cleanly by regarding $\mathcal{D}$-modules on the flag variety of $G_{\mathbb{R}}$ as deformation quantization of Lagrangian subvarieties of coadjoint orbits. The theory of $\mathcal{D}$-modules invented by Beilinson and Bernstein ([10]) has been proven to be a powerful tool for the study of representation theory. I relate it with Kirillov's orbit method [39] through nonabelian Hodge theory, which might shed a new light on the relationship between quantization and representation theory.

The third project is joint with Jonathan Block and Nigel Higson. We propose to develope a noncommutative version of the Oka principle in complex geometry to give a new simplified proof of the Connes-Kasparov Conjecture/Theorem, which is a special case of the deep Baum-Connes conjecture in the case of Lie groups.

## 2. Lie structure in algebraic geometry

2.1. **Analogy between Lie theory and algebraic geometry.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $k$ of characteristic zero. Then we have the Poincaré-Birkhoff-Witt (PBW) isomorphism

$$I_{PBW} : S\mathfrak{g} \to \mathcal{U}\mathfrak{g} \tag{2.1}$$

of $\mathfrak{g}$-modules, where $S\mathfrak{g}$ is the symmetric algebra of $\mathfrak{g}$ and $U\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$.

The PBW isomorphism is not in general an isomorphism of algebras. On the other hand, it was the fascinating discovery of Kirillov and Duflo [28] that the $\mathfrak{g}$-invariant part $(S\mathfrak{g})^{\mathfrak{g}}$ of $S\mathfrak{g}$ with its usual commutative product and the center $(\mathcal{U}\mathfrak{g})^{\mathfrak{g}}$ of $\mathcal{U}\mathfrak{g}$ are isomorphic via a 'twisted' PBW map. Namely, let $J \in \hat{S}\mathfrak{g}^*$ be the formal power series

$$J(x) = \det\left(\frac{1-e^{-ad_x}}{ad_x}\right), \tag{2.2}$$

where $ad : \mathfrak{g} \to End(\mathfrak{g})$ is the linear map defined by $ad_x(y) = [x,y]$ $(x, y \in \mathfrak{g})$ and hence $ad \in \mathfrak{g}^* \otimes End(\mathfrak{g})$ and $ad^n \in T^n\mathfrak{g}^* \otimes End(\mathfrak{g})$. We regard $ad^n$ as an element in $S^n\mathfrak{g}^* \otimes End(\mathfrak{g})$ through the projection $T^n\mathfrak{g}^* \to S^n\mathfrak{g}^*$. Then we can define the Duflo map

$$I_{Duflo} = I_{PBW} \circ J^{\frac{1}{2}},$$

which is an isomorphism of algebras $(S\mathfrak{g})^{\mathfrak{g}} \to (\mathcal{U}\mathfrak{g})^{\mathfrak{g}}$.

In the past twenty years people realized that there is a surprising analogue of Lie theory in the realm of algebraic geometry. Consider the diagonal embedding $\Delta : X \hookrightarrow X \times X$ of a smooth variety or complex manifold $X$. First of all, there is the Hochschild-Kostant-Rosenberg (HKR) isomorphism ([37], [22], [23])

$$I_{HKR} : HT^\bullet(X) := \bigoplus_{p+q=\bullet} H^p(X, \wedge^q TX) \to HH^\bullet(X), \tag{2.3}$$

where $HT^\bullet(X)$ is the cohomology of the sheaf of holomorphic polyvector fields on $X$, $HH^\bullet(X) = \mathrm{Ext}^\bullet_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ is the Hochschild cohomolgy of $X$ and $\mathcal{O}_\Delta$ is the derived direct image of the structure sheaf of $X$ via the diagonal embedding $\Delta : X \hookrightarrow X \times X$.

Both sides of the HKR isomorphism have natural sturctures of algebras. The exterior product of polyvector fields makes $HT^\bullet(X)$ into a graded commutative algebra, while the Hochschild cohomology as an Ext algebra has the Yoneda product. Unfortunately, the HKR map is not an algebra isomorphism in general. However, Kontsevich discovered that another map $I_{HKR} \circ Td_{TX}^{1/2}$ induces an isomorphism of graded algebras

$$I_{DKK} : HT^\bullet(X) \xrightarrow{\simeq} HH^\bullet(X), \tag{2.4}$$

where

$$Td_{TX} = \det\left(\frac{at_{TX}}{1-e^{-at_{TX}}}\right) \in \bigoplus_i H^i(X, \Omega^i) \tag{2.5}$$

is the Todd class of $TX$ and $Td_{TX}^{1/2}$ acts on the polyvector fields by contraction. Konsevich observed that his solution to the deformation quantization problem makes it possible to prove both the Duflo-Kirillov isomorphism and the isomorphism (2.4) in a unified way ([45], [21], [19]). A more generalized version involving both Hoschschild homology and cohomology was conjectured by Căldăraru ([23]) and was later proved in [20], which in particular implies the Riemann-Roch theorem in Dolbeault cohomology.

I aim to extend this deep connection between Lie theory and algebraic geometry to the case of general embeddings $i : X \hookrightarrow Y$ and study the algebraic structures on the Ext algebra

$$\mathrm{Ext}^\bullet_Y(i_*\mathcal{O}_X, i_*\mathcal{O}_X). \tag{2.6}$$

Such Ext algebras are related to open string states between D-branes in string theory (see, e.g., [43]). The key observation is that all the information we need is about the formal neighborhood $X^{(\infty)}_Y$ of $X$ inside $Y$.

$$\mathrm{Ext}^\bullet_Y(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathrm{Ext}^\bullet_{X'}(\mathcal{O}_X, \mathcal{O}_X),$$

where $X' = X^{(\infty)}_Y$ is the (completed) formal neighborhood of $X$ inside $Y$. Thus it is the first yet crucial step to understand the formal neighborhood.

2.2. **Dolbeault dga of a formal neighborhood.** In [65], I introduced the notion of the *Dolbeault dga* $(\mathcal{A}^\bullet(X^{(\infty)}_Y), \overline{\partial})$ of a closed embedding $i : X \hookrightarrow Y$ of complex manifolds. This dga is naturally defined without any auxilliary choices. In the case of the diagonal embedding $\Delta : X \hookrightarrow X \times X$, I showed in [66] that the Dolbeault dga is isomorphic to the Dolbeault complex $(\Omega^{0,\bullet}_X(\mathcal{J}^\infty_X), \overline{\partial})$ of the jet bundle $\mathcal{J}^\infty_X$ and provide a fine resolution of the structure sheaf of the formal space $X^{(\infty)}_Y$ ([65]). Hence the Dolbeault dga behaves exactly like the Dolbeault complex of an ordinary complex manifold (note that it is identical to the usual one when $X = Y$).

Just as any holomorphic vector bundle over a complex manifold is equivalent to a smooth vector bundle equipped with a flat $\overline{\partial}$-connection, holomorphic vector bundles and even coherent sheaves over $X^{(\infty)}_Y$ can be captured by modules over the Dolbeault dga $\mathcal{A}^\bullet(X^{(\infty)}_Y)$. Use the notion of the *perfect dg-category $\mathcal{P}_A$ of cohesive modules* over a dga $A$ introduced by Block [16], I was able to characterize the derived category of coherent sheaves over the formal neighborhood.

**Theorem 2.1** (Yu, [65])**.** *Let $i : X \hookrightarrow Y$ be a closed embedding of complex manifolds and $A = (\mathcal{A}^\bullet(X^{(\infty)}_Y), \overline{\partial})$ the Dolbeault dga of the formal neighborhood. Assume that $X$ is compact, then the homotopy category* Ho $\mathcal{P}_A$ *of the dg-category $\mathcal{P}_A$ is equivalent to $D^b_{coh}(X^{(\infty)}_Y)$.*

2.3. $L_\infty$**-algebroid.** The analogy described in the previous section is not a coincidence. In his work on Rozansky-Witten invariants [40], Kapranov realized that the shifted tangent vector bundle $TX[-1]$ of a complex manifold/algraic variety $X$ is naturally a Lie algebra object in the derived category $D^b(X)$ with the Lie bracket $TX[-1] \otimes TX[-1] \to TX[-1]$ given by the Atiyah class

$$\alpha_{TX} \in \mathrm{Ext}^1(S^2TX, TX)$$

of $TX$ ([4]). It was then observed by many people (e.g., [48], [53], [54]) that the universal enveloping algebra of $TX[-1]$ in $D^b(X)$ can be realized as $(\Delta^*\mathcal{O}_\Delta)^\vee$, whose sheaf cohomology $H^\bullet(X, (\Delta^*\mathcal{O}_\Delta)^\vee)$ is identified with the Hochschild cohomology $HH^\bullet(X)$ (the inverse image functor $\Delta^*$ here and all functors in due course are understood as derived functors.) In light of this, the HKR isomorphism (2.3) is the PBW isomorphism for $TX[-1]$ and the isomorphism $I_{DKK}$ (2.4) is the Dulo-Kirillov isomorphism for $TX[-1]$.

Moreover, Kapranov showed that the Dolbeault resolution $(\Omega^{0,\bullet-1}_X(TX), \overline{\partial})$ of $TX[-1]$ has an $L_\infty$-algebra structure, unique up to homotopy. Its (completed) Chevelley-Eilenberg (CE) complex is isomorphic to the dga $(\Omega_X^{0,\bullet}(\mathcal{J}_X^\infty), \overline{\partial})$. Namely, there is an isomorphism of dgas

$$(\Omega_X^{0,\bullet}(\mathcal{J}_X^\infty), \overline{\partial}) \simeq (\Omega_X^{0,\bullet}(\hat{S}(T^\vee X), \overline{\partial} + \alpha)), \tag{2.7}$$

where the differential of the dga on the right hand side is the usual $\overline{\partial}$ corrected by $\alpha$, which is given by the infinite sum of Atiyah class and all its higher covariant derivatives and corresponds exactly to the brackets of the $L_\infty$-structure on $TX[-1]$.

In my thesis [64], I constructed an isomorphism analogous to (2.7) for a general embedding

$$(\mathcal{A}^\bullet(X_Y^{(\infty)}), \overline{\partial}) \simeq (\Omega_X^{0,\bullet}(\hat{S}(N^\vee)), \mathfrak{D}), \tag{2.8}$$

where $N^\vee$ is the conormal bundle of the submanifold $X$. When the ambient manifold $Y$ is Kähler, I was able to write down an explicit formula for the differential $\mathfrak{D}$ on the RHS in terms of the Levi-Civita connection, the curvatures and the second fundamental form of the embedding.

Similar to the Koszul duality between the formal neighborhood of the diagonal embedding and the $L_\infty$-algebra $TX[-1]$, any isomorphism as in (2.8) also gives rise to an $L_\infty$-structure on (the Dolbeault resolution of) the shifted conormal bundle $N[-1]$ of the embedding. In general, however, the structure maps of this $L_\infty$-algebra are no longer $\mathcal{O}_X$-linear. Instead, one can read off an $\infty$-*anchor map* $\rho : N[-1] \to TX$ from the isomorphism (2.8) by restricting to the degree zero part $S^0 N^\vee$. The first component $\rho_1 : N[-1] \to TX$ gives rise to a cohomology class $[\rho_1] \in \mathrm{Ext}^1_X(N, TX)$, which is the obstruction to the holomorphic splitting of the normal exact sequence

$$0 \to TX \to TY|_X \to N \to 0. \tag{2.9}$$

Hence $N[-1]$ is an $L_\infty$-*algebroid*. Similar results in the algebraic context were established by Calaque, Căldăraru and Tu ([18]) during the same time when my work was done.

**Theorem 2.2** (Yu, [67])**.** *The shifted normal bundle $N[-1]$ of a closed embedding $i : X \hookrightarrow Y$ admits the structure of an $L_\infty$-algebroid, which is unique up to homotopy equivalence. The (completed) Chevalley-Eilenberg dga $(\mathcal{A}^\bullet_X(\hat{S}(N^\vee)), \mathfrak{D})$ of $N[-1]$ is isomorphic to the Dolbeault dga $\mathcal{A}^\bullet(X_Y^{(\infty)})$. Moreover, if $Y$ is Kähler, the structure maps of the $L_\infty$-algebroid $N[-1]$ can be constructed explicitly in terms of differential geometric quantities, such as the connection and curvature of $Y$ and the shape operator of the submanifold $X$.*

2.4. **Quantized analytic cycles and generalized Todd class.** Motivated by the work of Kashiwara and Schapira ([41], [42]), Grivaux defined in [31] for any $(X, \sigma)$ the *quantized cycle class* $q_\sigma(X)$ in $\bigoplus_{i=0}^d H^i(X, \wedge^i N^\vee)$ and proved in [30] the conjecture by Kashiwara that, for the diagonal embedding $X \hookrightarrow X \times X$, the quantized cycle class is exactly the Todd class of $X$ in $\bigoplus_{i=0}^{\dim X} H^i(X, \Omega^i_X)$, which provides a nice short proof of the Grothendieck-Riemann-Roch theorem in Dolbeault cohomology for complex manifolds. Therefore the quantized cycle class can be regarded as a generalization of the Todd class. Essentially, $q_\sigma(X)$ measures how the composition of isomorphisms

$$H^\bullet(X, \wedge^\bullet N) \xrightarrow{\sim} \mathrm{Ext}^\bullet_X(\mathcal{O}_X, i^! i_* \mathcal{O}_X) \xrightarrow{\sim} \mathrm{Ext}^\bullet_{X'}(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \xrightarrow{\sim} \mathrm{Ext}^\bullet_X(i^* i_* \mathcal{O}_X, \mathcal{O}_X) \xrightarrow{\sim} H^\bullet(X, \wedge^\bullet N)$$

deviates from being the identity map, where the first and the last isomorphisms are HKR-type maps and the second one is by the Grothendieck-Verdier duality. It remained as a question in Grivaux's paper [31] to compute $q_\sigma(X)$ in geometric terms for a general quantized cycle $(X, \sigma)$.

The Dolbeault dga of the formal neighborhood I have constructed and the dg-category of cohesive modules provide perfect tools to attack Grivaux's problem. There turns out to be an interesting cohomology class which serves as an obstruction for the quantized cycle class $q_\sigma(X)$ to look exactly like the usual Todd class. To describe this class, we first define the *transverse Atiyah class*

$$at^{\perp}_{X/Y} \in H^1(X, N \otimes S^2 N^{\vee}) \tag{2.10}$$

by pulling back the Atiayh class $\alpha_{TY} \in H^1(Y, TY \otimes S^2 T^{\vee} Y)$ of $TY$ to $X$ and applying the projection $TY \otimes S^2 TY^{\vee} \to N \otimes S^2 N^{\vee}$ via the splitting $\sigma$. There is also a related class $at^{\top}_{X/Y} \in H^1(X, TX \otimes S^2 N^{\vee})$ defined via the projection $TY \otimes S^2 T^{\vee} Y \to TX \otimes S^2 N^{\vee}$. Composing $at^{\top}_{X/Y}$ with the Atiyah class $at_N \in H^1(X, TX^{\vee} \otimes End(N))$ of the normal bundle, we get a class $\gamma \in H^2(X, (N^{\vee})^{\otimes 3} \otimes N)$.

In [68], I showed that, when the class $\gamma$ vanishes, the quantized cycle class $q_\sigma(X)$ can be expressed by the same power series $x/(1-e^{-x})$ in the definition of Todd class with $x$ replaced by $at^{\perp}_{X/Y}$ and hence provides an answer to Grivaux's question. I plan to explain my results in terms of derived geometry and exploit its relationship with the work of Grady and Gwilliam ([26]) where they gave a proof of algebraic index theorem using Costello's homological quantum field theory ([27]).

## 3. Geometric representation theory

3.1. **Mackey analogy.** Inspired by the concept from physics of the contraction of a Lie group to a Lie subgroup ([38]), Mackey suggested in 1975 ([47]) that there should be a correspondence between "almost" all the irreducible unitary representations of a noncompact semisimple group $G_{\mathbb{R}}$ and the irreducible unitary representations of its contraction to a maximal compact subgroup $K_{\mathbb{R}}$. The contraction group is defined to be the group

$$G_{\mathbb{R},0} := K_{\mathbb{R}} \ltimes \mathfrak{g}_{\mathbb{R}}/\mathfrak{k}_{\mathbb{R}},$$

where $\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}(G_{\mathbb{R}})$ and $\mathfrak{k}_{\mathbb{R}} = \mathrm{Lie}(K_{\mathbb{R}})$ are the corresponding Lie algebras and $\mathfrak{g}_{\mathbb{R}}/\mathfrak{k}_{\mathbb{R}}$ is regarded as an abelian group with the usual addition of vectors. The group $G_{\mathbb{R},0}$ is called the *Cartan motion group* of $G_{\mathbb{R}}$. It is a surprising analogy since the algebraic structures of the groups $G_{\mathbb{R}}$ adn $G_{\mathbb{R},0}$ are quite different. While the representation theory of the semisimple group $G_{\mathbb{R}}$ is rather complicated and even decades after Mackey, the problem of effective discription of the unitary dual $\widehat{G_{\mathbb{R}}}$ of such $G_{\mathbb{R}}$ is not fully solved yet. On the other hand, Mackey himself developed a full theory of representations of semidirect product groups like $G_{\mathbb{R},0}$, so the unitary dual of $G_{\mathbb{R},0}$ is much easier to describe. If the analogy holds in general, it might provide a new approach to the construction of unitary representations of $G_{\mathbb{R}}$.

Mackey himself was very cautious even after making a number of calculations in support of his conjecture. One reason is that, even if the analogy holds for parameter spaces, it seems to behave poorly at the level of representation spaces. For instance, while there are unitary unitary irreducible representations of $G_{\mathbb{R},0}$ whose underlying vector spaces are of finite dimensions (on which the $\mathfrak{g}_{\mathbb{R}}/\mathfrak{k}_{\mathbb{R}}$ part of $G_{\mathbb{R},0}$ acts trivially), all nontrivial unitary representations of $G_{\mathbb{R}}$ are infinite-dimensional.

However, Connes later pointed out that there is a connection between the Mackey analogy and the Connes-Kasparov conjecture in $C^*$-algebra K-theory ([5]), which suggests that the reduced dual, or equivalently, the tempered dual of $G_{\mathbb{R}}$ should correspond to the unitary dual of $G_{\mathbb{R},0}$, at least K-theoretically. Following Connes' insight, Higson suggested that the correspondence ought to be exactly a set theoretical bijection. In other words, Mackey analogy can be thought of as a stronger version of the Connes-Kasparov conjecture. In his paper [34], Higson examined the case where $G_{\mathbb{R}}$ is a connected complex semisimple group (regarded as a real group) and showed that there is a natural bijection between the reduced duals of $G_{\mathbb{R}}$ and $G_{\mathbb{R},0}$, with the already known classification of irreducible tempered representations on both sides in hand. Later in [35], he strengthened this result by showing that there is even a natural bijection between the admissible duals of $G_{\mathbb{R}}$ and $G_{\mathbb{R},0}$ when $G_{\mathbb{R}}$ is a complex group.

Whether there is a Mackey bijection between admissble duals when $G_{\mathbb{R}}$ is a real group has remained unsolved for a long time. Recently, Afgoustidis has established a very clean and natural bijection between the tempered dual of $G_{\mathbb{R}}$ and the unitary dual of $G_{\mathbb{R},0}$ using the Knapp-Zuckerman classification of tempered irreducible representations of $G_{\mathbb{R}}$ ([1], [44]), which will be summarized below. This bijection is in particular an extension of Vogan's bijection between irreducible tempered representation of $G_{\mathbb{R}}$ with real infinitesimal characters with their unique minimal $K$-types.

**Theorem 3.1** (Vogan, [62])**.** *Any irreducible tempered representations of a real reductive Lie group $G_{\mathbb{R}}$ with real infinitesimal character has a unique $K_{\mathbb{R}}$-type. This gives a bijection between the equivalence classes of irreducible tempered representations of a real reductive Lie group $G_{\mathbb{R}}$ with real infinitesimal characters and the equivalence classes of irreducible representations of its maximal compact subgroup $K_{\mathbb{R}}$.*

Afgoustidis also studied the analogy at the level of representation spaces, writing down explict contractions from representations of $G_{\mathbb{R}}$ to that of $G_{\mathbb{R},0}$ in the case of spherical principal series representations, discrete series and limit of discrete series representations. However, it is not clear yet if there is a general way to construct such contractions for all tempered representations, even for those with real infinitesimal characters.

I study the Mackey analogy from the perspective of geometric representation theory. My construction works in general for any admissible representations of $G_{\mathbb{R}}$. It also naturally explains the 'finite-dimensional to infinite-dimensional" phenomenon at the level of representation spaces. My approach is also related to deformation quantization, nonabelian Hodge theory and Kirillov's coadjoint orbit method. I will summerize my method below.

3.2. **Mackey correspondence.** We briefly describe Afgoustidis's Mackey correspondence for tempered representations at the level of parameter spaces. Suppose $G_{\mathbb{R}}$ is a connected real semisimple Lie group with finite center. Fix a maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$. We denote the complexification of $G_{\mathbb{R}}$ by $G = G_{\mathbb{C}}$ and its Lie algebra by $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Denote by $K = K_{\mathbb{C}}$ the complexification of $K_{\mathbb{R}}$ and its Lie algebra by $\mathfrak{k} = \mathfrak{k}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Let $\sigma_{\mathbb{R}}$ and $\sigma = \sigma_{\mathbb{C}}$ be the corresponding Cartan involutions of $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}$ respectively, which have $\mathfrak{k}_{\mathbb{R}}$ and $\mathfrak{k}$ as fixed points respectively. Let $\mathfrak{p}_{\mathbb{R}}$ and $\mathfrak{p} = \mathfrak{p}_{\mathbb{C}}$ be the $-1$-eigenspaces of the involutions $\sigma_{\mathbb{R}}$ and $\sigma$ respectively, so we have $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The motion group $G_{\mathbb{R},0} = K_{\mathbb{R}} \ltimes \mathfrak{p}_{\mathbb{R}}$ has Lie algebra $\mathfrak{g}_{\mathbb{R},0} = \mathfrak{k}_{\mathbb{R}} \ltimes \mathfrak{p}_{\mathbb{R}}$, whose complexification is $\mathfrak{g}_0 = \mathfrak{k} \ltimes \mathfrak{p}$.

Suppose $\mathfrak{a}_{\mathbb{R}}$ is a maximal abelian subalgebra of $\mathfrak{p}_{\mathbb{R}}$ and $\mathfrak{a}$ is its complexification. Let $W_{\mathfrak{a}}$ be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$. We consider $K_{\mathbb{R}}$-orbits in $\mathfrak{p}_{\mathbb{R}}^*$. Any $K_{\mathbb{R}}$-orbit in $\mathfrak{p}_{\mathbb{R}}^*$ intersect with $\mathfrak{a}_{\mathbb{R}}^*$ at a unique regular $W_{\mathfrak{a}}$-orbit, where we identify $\mathfrak{a}_{\mathbb{R}}^*$ as a subspace of $\mathfrak{p}_{\mathbb{R}}^*$ using the Killing form. Hence choosing a $K$-orbit of $\mathfrak{p}_{\mathbb{R}}^*$ is equivalent to choosing a $\chi \in \mathfrak{a}^*$ up to a $W_{\mathfrak{a}}$-symmetry. Denote the stabilizer of $\chi$ in $K_{\mathbb{R}}$ by $K_{\mathbb{R},\chi}$. According to Afgoustidis, a *Mackey datum* is a pair $(\chi, \mu)$ in which $\mu$ is an irreducible unitary representation of $K_{\chi}$. Given such a datum, we can produce a unitary representation $M_0(\delta)$ of $G_{\mathbb{R},0}$ by induction:

$$M_0(\delta) := \mathrm{Ind}_{K_{\mathbb{R},\chi} \ltimes \mathfrak{p}_{\mathbb{R}}}^{G_0} \left[ \mu \otimes e^{i\chi} \right].$$

Mackey showed that all such $M_0(\delta)$ give a complete list of irreducible unitary representations of $G_{\mathbb{R},0}$. Moreover, two Mackey data $\delta_1 = (\chi_1, \mu_1)$ and $\delta_2 = (\chi_2, \mu_2)$ give rise to unitarily equivalent representations if and only if there is an element of the Weyl group $W_{\mathfrak{a}}$ which sends $\chi_1$ to $\chi_2$ and

$\mu_1$ to an irreducible $K_{\chi_2}$-representations which is unitarily equivalent with $\mu_2$. We say that the Mackey data $\delta_1$ and $\delta_2$ are equivalent.

Now we define the Afgoustidis's Mackey bijection from $\widehat{G_{\mathbb{R},0}} \to \widehat{G_{\mathbb{R}}}$. Fix a maximal torus $T$ of $K_{\mathbb{R}}$. Given a Mackey datum $\delta = (\chi, \mu)$, set $T_\chi = K_{\mathbb{R},\chi} \cap T$. It is a maximal torus of $K_{\mathbb{R},\chi}$. Take the centralizer $\mathfrak{a}_\chi$ of the Lie algebra $\mathfrak{t}_\chi$ of $T_\chi$ in $\mathfrak{a}$ and from the vector subgroup $A_\chi := \exp_G(\mathfrak{a}_\chi)$. The abelian subgroup $H_\chi := T_\chi A_\chi$ is then a Cartan subgroup of $G_{\mathbb{R}}$. Let $L_\chi$ be the centrailizer of $A_\chi$ in $G_{\mathbb{R}}$ and we have $L_\chi = M_\chi A_\chi$, where $M_\chi$ is a reductive subgroup of $G_{\mathbb{R}}$. Choose a system $\Delta^+$ of postive roots for the pair $(\mathfrak{g}, \mathfrak{h})$ and define the subalgebra $\mathfrak{n}_\chi$ of $\mathfrak{g}_{\mathbb{R}}$ as the real part of the sum of root spaces in $\mathfrak{g}$ for those positive roots which do not vanish on $\mathfrak{a}_\chi$. Set $N_\chi := \exp_{G_{\mathbb{R}}}(\mathfrak{n}_\chi)$ and $P_\chi := M_\chi A_\chi N_\chi$. Then $P_\chi$ is a cuspidal parabolic subgroup.

The $K_{\mathbb{R},\chi}$-representation $\mu$ coming with the Mackey datum $\delta$ determines a tempered representation $V_{M_\chi}(\mu)$ of $M_\chi$ by Vogan's theorem. $\chi$ determines a one-dimensional unitary rerpresentation $e^{i\chi}$ of $A_\chi$ and it extends to a representation of $A_\chi N_\chi$ where $N_\chi$ acts trivially. The work of Harish-Chandra, Knapp and Zuckerman showed that the induced unitary representation

$$M(\delta) := \mathrm{Ind}_{P_\chi}^{G_{\mathbb{R}}} \left[ V_{M_\chi}(\mu) \otimes e^{i\chi} \right]$$

of $G$ is irreducible and tempered. Moreover, $M(\delta_1)$ and $M(\delta_2)$ are uitarily equivalent if and only if $\delta_1$ and $\delta_2$ are equivalent as Mackey data. Therefore the tempered dual of $G_{\mathbb{R}}$ and the unitary dual of $G_{\mathbb{R},0}$ are both parametrized by Mackey data.

3.3. **$\mathcal{D}$-modules.** Let us recall the construction of twisted $\mathcal{D}$-modules on the flag variety. Let $X$ be the flag variety of $G$, which is the variety of all Borel subalgeras $\mathfrak{b}$ in $\mathfrak{g}$. Let $\mathfrak{g}^\circ = \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$ be the sheaf of local sections of the trivial bundle $X \times \mathfrak{g}$. Let $\mathfrak{b}^\circ$ be the vector bundle on $X$ whose fiber $\mathfrak{b}_x$ at any point $x$ of $X$ is the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ corresponding to $x$. Similarly, let $\mathfrak{n}^\circ$ be the vector bundle whose fiber $\mathfrak{n}_x$ is the nilpotent ideal $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$ of the corresponding Borel subalgebra $\mathfrak{b}$. $\mathfrak{b}^\circ$ and $\mathfrak{n}^\circ$ can be considered a subsheaves of $\mathfrak{g}^\circ$. $\mathfrak{g}^\circ$ has a natural structure of Lie algebroid: the differential of the action of $G$ on $X$ defines a natural map from $\mathfrak{g}$ to the tangent bundle $TX$ of $X$ and hence induces an anchor map $\tau : \mathfrak{g} \to TX$. The Lie structure on $\mathfrak{g}$ is given by

$$[f \otimes \xi, g \otimes \eta] = f\tau(\xi)g \otimes \eta - g\tau(\eta)f \otimes \xi + fg \otimes [\xi, \eta]$$

for any $f, g \in \mathcal{O}_X$ and $\xi, \eta \in \mathfrak{g}$. The kernel of $\tau$ is exactly $\mathfrak{b}^\circ$, so $\mathfrak{b}^\circ$ and $\mathfrak{n}^\circ$ are sheaves of Lie ideals in $\mathfrak{g}^\circ$.

We then form the universal enveloping algebra of the Lie algebroid $\mathfrak{g}^\circ$, which is the sheaf $\mathcal{U}\mathfrak{g}^\circ = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{U}\mathfrak{g}$ of associative algebras. The sheaf of left ideals $\mathcal{U}\mathfrak{g}^\circ\mathfrak{n}^\circ$ generated by $\mathfrak{n}^\circ$ in $\mathcal{U}\mathfrak{g}^\circ$ is a sheaf of two-sided ideals in $\mathcal{U}\mathfrak{g}^\circ$, hence the quotient $\mathcal{D}_{\mathfrak{h}} = \mathcal{U}\mathfrak{g}^\circ / \mathcal{U}\mathfrak{g}^\circ\mathfrak{n}^\circ$ is a sheaf of associative algebras on $X$. The natural morphism from $\mathfrak{g}^\circ$ to $\mathcal{D}_{\mathfrak{h}}$ induces an inclusion of $\mathfrak{h}^\circ = \mathfrak{b}^\circ/\mathfrak{n}^\circ$ into $\mathcal{D}_{\mathfrak{h}}$. $\mathfrak{h}^\circ$ turns out to be a trivial vector bundle and its global sections over $X$ is the *abstract Cartan algebra* $\mathfrak{h}$ of $\mathfrak{g}$, which is independent of the choice of its embedding into borel subalgebras. Moroever, $\mathcal{U}\mathfrak{h} = \Gamma(X, \mathcal{D}_{\mathfrak{h}})^G$. For any $\lambda \in \mathfrak{h}^*$, set $\mathcal{D}_\lambda = \mathcal{D}_{\mathfrak{h}} \otimes_{\mathcal{U}\mathfrak{h}} \mathbb{C}_{\lambda+\rho}$, which is a sheaf of twisted differential operators. Then the Beilinson-Bernstein localization theorem ([10]) says that the category of quasicoherent sheaf of $\mathcal{D}_\lambda$-modules is equivalent to the category of $\mathcal{U}\mathfrak{g}$-modules with infinitesimal character $\lambda$ by taking global sections.

Under the Beilinson-Bernstein equivalence, $(\mathfrak{g}, K)$-modules correspond to Harish-Chandra sheaves, which are $K$-equivariant holonomic $\mathcal{D}_\lambda$-modules. The standard way to produce an irreducible Harish-Chandra sheaf is to take a $K$-orbit $Q$ in $X$ together with an irreducible $K$-homogeneous connection $\phi$ on $Q$, which satisfies certain compatibility condition with $\lambda$, and then push forward

$\phi$ to a $\mathcal{D}_\lambda$-module $\mathcal{I}(Q,\phi)$ on $X$, called the standard Harish-Chandra sheaf attached to $(Q,\phi)$. For generic $\lambda$, $I(Q,\phi)$ is irreducible and produces an irreducible $(\mathfrak{g},K)$-module, otherwise it contains a unique irreducible subsheaf of $\mathcal{D}_\lambda$-modules, denoted by $\mathcal{L}(Q,\phi)$. With certain assumptions on $\lambda$ and $Q$, this gives a geometric classification of the admissible representations of $G_{\mathbb{R}}$.

3.4. **"Localization" of $G_{\mathbb{R},0}$-representations.** I am going to describe how to realize $(\mathfrak{g}_0,K)$-modules as geometric objects on the flag variety $X$ and how twisted $\mathcal{D}$-modules deform to them. Since I will consider the entire admissible duals of both groups, the Lie algebras involved here are all complex. First of all, $\mathfrak{g}_0$ and $\mathfrak{g}$ fit into a continous family of Lie algebras $\mathfrak{g}_t$, $t\in\mathbb{C}$, with the fiber at $t=0$ being the Lie algebra $\mathfrak{g}_0$ and other fibers $\mathfrak{g}_t$, $t\neq 0$, isomorphic to $\mathfrak{g}$. A convenient way to describe it is as follows: take the trivial vector bundle $\mathbb{C}\times\mathfrak{g}$ and regard it as a sheaf of $\mathcal{O}_{\mathbb{C}}$-modules over the affine line $\mathbb{C}$. It is a sheaf of Lie algebras and its module of global sections is $\mathfrak{g}[t]=\mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[t]$, where $t$ is the coordinate function of $\mathbb{C}$. Then we can think of $\mathfrak{g}_t$ as the subsheaf in $\mathbb{C}\times\mathfrak{g}$ of germs sections which take values in $\mathfrak{k}\subset\mathfrak{g}$ at $0\in\mathbb{C}$. In other words, $\mathfrak{g}_t=\mathfrak{k}\oplus t\mathfrak{p}$. This is a sheaf of Lie subalgebras.

We can extend this construction the flag variety $X$ and form the trivial vector bundle $\mathfrak{g}^\circ[t]=X\times\mathbb{C}\times\mathfrak{g}$ and its subsheaf $\mathfrak{g}_t^\circ=\mathfrak{k}[t]\oplus t\mathfrak{p}^\circ[t]$ of germs of local sections which take values in the trivial vector bundle $\mathfrak{k}^\circ=X\times\mathfrak{k}$ over $X\times\{0\}$. Both are Lie algebroids over $X\times\mathbb{C}$ whose anchor maps are induced by the group actions on $X$ and take values in $TX[t]=TX\otimes_{\mathbb{C}}\mathbb{C}[t]$. $\mathfrak{g}_t^\circ$ is a subsheaf of Lie ideals of $\mathfrak{g}^\circ[t]$. Global sections of $\mathfrak{g}_t^\circ$ is exactly $\mathfrak{g}_t$. Similarly, we form the pullback $\mathfrak{b}^\circ$ and $\mathfrak{n}^\circ$ to vector bundles $\mathfrak{b}^\circ[t]$ and $\mathfrak{n}^\circ[t]$ over $X\times\mathbb{C}$ respectively. We form the sheaf of universal enveloping algebras $\mathcal{U}(\mathfrak{g}^\circ[t])$ and the sheaf of subalgebras $\mathcal{U}(\mathfrak{g}_t^\circ)$. Analogous to the case of one single $X$, we have the subsheaf of two-sided ideals $\mathcal{I}_{\mathfrak{n}}=\mathcal{U}(\mathfrak{g}^\circ[t])\mathfrak{n}^\circ[t]$ of $\mathcal{U}(\mathfrak{g}^\circ[t])$. Similar to the construction of the sheaf $\mathcal{D}_{\mathfrak{h}}$, we define $\mathcal{D}_{\mathfrak{h},t}:=\mathcal{U}\mathfrak{g}_t^\circ/(\mathcal{U}\mathfrak{g}_t^\circ\cap\mathcal{I}_{\mathfrak{n}})$. Note that this is no longer a local free sheaf over $X\times\mathbb{C}$. The restriction of $\mathcal{D}_{\mathfrak{h},t}$ to $X\times\{t\}$ for $t\neq 0$ is isomorphic to $\mathcal{D}_{\mathfrak{h}}$, yet its restriction $\mathcal{D}_{\mathfrak{h},t}/t\mathcal{D}_{\mathfrak{h},t}$ to $t=0$ is not locally free since the intersection of $\mathfrak{k}^\circ[t]\subset\mathfrak{g}^\circ[t]$ with $\mathfrak{n}^\circ[t]$ is not. However, those sheaves locally free when restricted to $K$-orbits of $X$, as we will see below.

Now assume a $K$-orbit $Q$ in $X$ is given. Denote the embedding by $i:Q\hookrightarrow X$. I am going to construct an analogue of the transfer bimodule used to form direct images of usual twisted $\mathcal{D}$-modules. Set

$$\mathcal{D}_{\mathfrak{h},Q}:=i^{-1}(\mathcal{D}_{\mathfrak{h},t})\otimes_{i^{-1}\mathcal{O}_Q}\omega_{Q/X}$$

where $\omega_{Q/X}$ is the relative canonical bundle of $Q$ in $X$. It is a left $i^{-1}(\mathcal{D}_{\mathfrak{h},t})$-module. The restriction of $\mathfrak{k}^\circ[t]$ to the $K$-orbit $Q$ is still a Lie algebroid, even though $\mathfrak{g}^\circ$ no longer is, and $\mathcal{U}(\mathfrak{k}^\circ[t])$ acts on $\mathcal{D}_{\mathfrak{h},Q}$ from right.

We want to construct the analogue of $\mathcal{D}_\lambda$ for $Q\times\mathbb{C}$. We still write $\mathfrak{g}^\circ$, $\mathfrak{g}_t^\circ$, $\mathfrak{b}^\circ$, $\mathfrak{n}^\circ$ for their restrictions to $Q$. Set $\mathfrak{b}_t^\circ:=\mathfrak{g}_t^\circ\cap\mathfrak{b}^\circ[t]$ and $\mathfrak{n}_t^\circ:=\mathfrak{g}_t^\circ\cap\mathfrak{n}^\circ[t]$. Since over the $K$-orbit the sheaves $\mathfrak{b}^\circ\cap\mathfrak{k}^\circ$ and $\mathfrak{n}^\circ\cap\mathfrak{k}^\circ$ have geometric fibers with constant ranks and hence are locally free, $\mathfrak{b}_t^\circ$ and $\mathfrak{n}_t^\circ$ are also locally free over $Q\times\mathbb{C}$. The fiber $\mathfrak{b}_{0,x}$ of the restriction $\mathfrak{b}_0^\circ=\mathfrak{b}_t^\circ|_{Q\times\{0\}}$ can be thought of as the contraction of the Borel subalgebra $\mathfrak{b}_x$ to a subalgebra of $\mathfrak{g}_0$. Analogously for $\mathfrak{n}_0^\circ$. We define $\mathfrak{h}_{Q,t}^\circ=\mathfrak{b}_t^\circ/\mathfrak{n}_t^\circ$. It is canonically isomorphic to the trivial vector bundle $\mathfrak{h}^\circ|_Q[t]$ over $Q\times\mathbb{C}$. The choice of the $K$-orbit $Q$ determines an involution $\sigma_Q$ on the abstract Cartan subalgebra $\mathfrak{h}$, which gives rise to a decomposition $\mathfrak{h}=\mathfrak{t}_Q\oplus\mathfrak{a}_Q$. We have a natural isomorphism $m_t:\mathfrak{h}^\circ[t]=\mathfrak{t}_Q^\circ[t]\oplus\mathfrak{a}_Q^\circ[t]\to\mathfrak{h}_Q^\circ[t]=\mathfrak{t}_Q^\circ[t]\oplus t\mathfrak{a}_Q^\circ[t]$ by multiplication the $\mathfrak{a}_Q^\circ[t]$ part by $t$.

One can show that there is an isomorphism $\mathcal{D}_{\mathfrak{h},Q}\simeq(\mathcal{U}\mathfrak{g}_t^\circ/\mathcal{U}\mathfrak{g}_t^\circ\mathfrak{n}_t^\circ)\otimes_{\mathcal{O}_Q}\omega_{Q/X}$, where we only use the pointwise multiplication of the bundle $\mathcal{U}\mathfrak{g}_t^\circ$. Then we have $\mathfrak{h}_{Q,t}^\circ$ acting on $\mathcal{D}_{\mathfrak{h},Q}$ from the right and so is $\mathcal{U}\mathfrak{h}_{Q,t}^\circ$. For a given $\lambda\in\mathfrak{h}^*$, we can take any $\lambda(t)\in\mathfrak{h}^*[t]$, in particular, $\lambda(t)=\lambda+\rho|_{\mathfrak{t}_Q}+t\rho|_{\mathfrak{a}_Q}$.

This determines a character $\lambda_t := \lambda(t) \circ (m_t^{-1}) = \lambda|_{\mathfrak{t}_Q} + (\lambda|_{\mathfrak{a}_Q}/t) + \rho$ of $\mathfrak{h}_{Q,t}$ (written as an element in $\mathfrak{t}_Q^*[t] \oplus t^{-1}\mathfrak{a}_Q^*[t]$). We then set

$$\mathcal{D}_Q^{\lambda(t)} := \mathcal{D}_{\mathfrak{h},Q} \otimes_{\mathcal{U}\mathfrak{h}_{Q,t}^\circ} (\mathbb{C}[t])_{\lambda_t}.$$

$\mathcal{D}_Q^{\lambda(t)}$ inherits a right $\mathcal{U}(\mathfrak{k}^\circ[t])$-module structure from that of $\mathcal{D}_{\mathfrak{h},Q}$. The specialization of $\mathcal{D}_Q^{\lambda(t)}$ to any $t \neq 0$ is isomorphic to $\mathcal{D}_{X \leftarrow Q}^{\lambda_t} = (i^{-1}\mathcal{D}_{\lambda_t}) \otimes \omega_{Q/X}$ on $Q$, which is exactly the transfer bimodule used to define the direct image functor of twisted $\mathcal{D}$-modules.

Now for any $K$-homogeneous connection $\phi$ on $Q$ which is compatible with $\lambda + \rho$ restricted to $\mathfrak{t}_Q$, we set

$$\mathcal{I}(\lambda(t), Q, \phi) = Ri_* \left( \mathcal{D}_Q^{\lambda(t)} \otimes_{\mathcal{U}\mathfrak{k}^\circ} \phi \right).$$

It is a left $\mathcal{D}_{\mathfrak{h},t}$-module. The specialization of $\mathcal{I}(\lambda(t), Q, \phi)$ to $t = 1$ is a $\mathcal{D}_\lambda$-module over $X$, hence its global sections give a $(\mathfrak{g}, K)$-module with infinitesimal character $\lambda_1 = \lambda$. When $\lambda$ is regular and antidominant, it has a unique irreducible submodule, denoted by $M(\lambda, Q, \phi)$. The global sections of the specialization $\mathcal{I}_0(\lambda, Q, \phi) := \mathcal{I}(\lambda(t), Q, \phi)|_{t=0}$ give a $(\mathfrak{g}_0, K)$-module, denoted by $\tilde{M}_0(\lambda, Q, \phi)$ . For generic values of $\lambda$ it is irreducible. In general, let $\chi = \lambda|_{\mathfrak{a}_Q}$ and regard $\chi$ as an element in $\mathfrak{p}^*$, then the Lie algebra of its stabilizer $K_\chi$ in $K$ has Cartan subalgebras isomorphic to $\mathfrak{t}_Q$. The restriction of $\lambda + \rho$ to $\mathfrak{t}_Q$ determines a irreducible representation $\mu$ of $K_\chi$ (or $K_{\mathbb{R},\chi}$). We therefore get a Mackey data $\delta = (\chi, \mu)$ and the associated representation $M_0(\delta)$ of $G_{\mathbb{R},0}$, when the data considered are all real.

**Theorem 3.2.** *The $(\mathfrak{g}_0, K)$-module $\tilde{M}_0(\lambda, Q, \phi)$ has a unique irreducible quotient, denoted by $M_0(\lambda, Q, \phi)$, which is isomorphic to the Harish-Chandra module of $M_0(\delta)$.*

In [50] Mirkovic showed that $(\mathfrak{g}, K)$-modules of the form $M(\lambda, Q, \phi)$ give all (the Harish-Chandra modules of) the tempered representations of $G_\mathbb{R}$ with certain assumptions on the triple $(\lambda, Q, \phi)$. Hence we have,

**Theorem 3.3.** *The correspondence $M(\lambda, Q, \phi) \longleftrightarrow M_0(\lambda, Q, \phi)$ coincides with Afgoustidis's Mackey correspondence if we restrict to tempered representations of $G_\mathbb{R}$ and unitary representations of $G_0$.*

In my approach, it is natural to expect that

**Conjecture 3.4.** *The Mackey bijection can be extended to the entire admissible duals of $G$ and $G_0$.*

3.5. **Deformation quantization of twisted characteristic cycles.** In fact, there is geometric meaning for the sheaf $\mathcal{I}_0(\lambda, Q, \phi)$ above: $\lambda|_{\mathfrak{a}_Q}$ considered as an element in $\mathfrak{g}^*$ determines a $G$-coadjoint $O_\lambda$ in $\mathfrak{g}^*$, which is a twisted cotangent bundle over the flag variety $X$. The intersection $Y_\lambda = O_\lambda \cap \mathfrak{p}^*$ is a $K$-equivariant affine variety, which for generic $\lambda$ is a single smooth closed $K$-orbit. The stabilizer of $\lambda|_{\mathfrak{a}_Q}$ as a point in $Y_\lambda$ is exactly the complexification of $K_\chi$, and hence the $K_\chi$ representation $\mu$ determines a $K$-equivariant vector bundle $\mathcal{L}_\mu$ over $O_\lambda$. Its direct image under the projection $O_\lambda \to X$ is exactly the sheaf $\mathcal{I}_0(\lambda, Q, \phi)$. Moreover, $Y_\lambda$ is Lagrangian subvariety of $O_\lambda$ with respect to the holomorphic Kirillov-Kostant-Souriau (KKS) symplectic form.

The pair $(Y_\lambda, \mathcal{L}_\mu)$ is similar to the notion of characteristic cycles of representations ([55]). While the usual characteristic cycle is always a Lagrangian subvariety of the cotangent bundle of the flag variety $X$ so it loses the information about the infinitesimal character of the representation, $Y_\lambda$ is a subvariety of a twisted cotangent bundle of $X$ and together with the bundle $\mathcal{L}_\mu$ they keep enough information to recover the representation of $G_\mathbb{R}$.

3.6. **The example of $SL(2,\mathbb{R})$.** Let me illustrate the general framework above in the case of $SL(2,\mathbb{R})$. Here we take $G_\mathbb{R} = SU(1,1)$ since $SU(1,1)$ is conjugate to $SL(2,\mathbb{R})$ inside $G = SL(2,\mathbb{C})$ and the formulas have simpler appearance. The flag variety is $X = \mathbb{P}^1$. We denote by $\theta \in \mathfrak{h}^*$ the positive root of $\mathfrak{g}$ and hence $\rho = \frac{1}{2}\theta$. We put $c = \check{\theta}\ (\lambda)$, where $\check{\theta} \in \mathfrak{h}$ is the dual root of $\theta$. The Lie algebra

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \middle| \ \alpha, \beta, \gamma \in \mathbb{C} \right\}$$

has the standard basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfying the standard commutation relations

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h. \tag{3.1}$$

We define the coordinate $z$ on $\mathbb{P}^1 - \{\infty\}$ by $z([x_0 : 1]) = x_0$ to identify $\mathbb{P}^1 - \{\infty\}$ with the complex plane $\mathbb{C}$. Then we get a trivialization of $\mathcal{D}_\lambda$ in which

$$e = -\partial_z, \quad f = z^2\partial_z + (c+1)z, \quad h = -2z\partial_z - (c+1).$$

We obtain a second trivialization on the open $K$-orbit $\mathbb{C}^* \subset \mathbb{C}$ by the automorphism of $\mathcal{D}_{\mathbb{C}^*}$ induced by

$$\partial_z \mapsto \partial_z - \frac{c+1}{2z} = z^{\frac{c+1}{2}} \partial_z z^{-\frac{c+1}{2}}$$

which satisfies

$$e = -\partial_z + \frac{c+1}{2z}, \quad f = z^2\partial_z + \frac{c+1}{2}z, \quad h = -2z\partial_z.$$

In other words, The trivialization is via the canonical isomorphism $\mathcal{U}\mathfrak{k}^\circ|_Q \simeq \mathcal{D}_\lambda|_Q$ induced by the composition $\mathcal{U}\mathfrak{k}^\circ \hookrightarrow \mathcal{D}_\mathfrak{h} \twoheadrightarrow \mathcal{D}_\lambda$ (since the stabilizer of the $K$-action on $Q$ is discrete in this case), so that the expression for $H$ does not change when $\lambda$ varies. The open $K$-orbit is related to principal series representations of $G_\mathbb{R}$. As indicated above, we replace $c$ by $c/t$ to get the family. Equivalently, set

$$e_t = t\left(-\partial_z + \frac{1}{2z}\right) + \frac{c}{2z}, \quad f_t = t\left(z^2\partial_z + \frac{1}{2}z\right) + \frac{c}{2}z, \quad h_t = h = -2z\partial_z,$$

and the triple $(e_t, f_t, h_t)$ still satisfies the commutation relation (3.1) when $t \neq 0$. When $t = 0$, $(e_0, f_0, h_0)$ forms a basis of the Lie algebra $\mathfrak{g}_0$ of the motion group. Moreover, the $S(\mathfrak{p})$-module structure is determined by the relation $e_0 f_0 = c^2/4$, so the Lagrangian subvariety correponding to principal series representations with infinitesimal character $\lambda$ is the intersection of $\mathfrak{p}$ with the coadjoint orbit in $\mathfrak{g}$ determined by the equation

$$\alpha^2 + \beta\gamma = \frac{c^2}{4},$$

where we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the Killing form.

3.7. **Extended Kostant-Sekiguchi-Vergne correspondence and Matsuki correspondence.** The nonabelian Hodge theory ([57]) gives a bijection between the moduli space of flat connections over a punctured Riemann surface and the moduli space of Higgs bundles. It specializes to the Nahm's equations, which was used by Vergne to establish the Kostant-Sekiguchi-Vergne (KSV) correspondence and gave diffeomorphisms between the nilpotent $G_\mathbb{R}$-orbits in $\mathfrak{g}_\mathbb{R}^*$ and the nilpotent $K$-orbits in $\mathfrak{p}^*$ ([56], [61]). Later Bielawski and Biquard extended it to a correspondence between general coadjoint orbits in $\mathfrak{g}_{\mathbb{R}}$ and $K$-orbits in $\mathfrak{p}^*$ ([12], [13]). I plan to show that this extend the correspondence such that it is compatible with the Mackey correspondence. Namely, any repesentation $V_0$ of $G_{\mathbb{R},0}$ is determined by a $K$-orbit and an $K$-equivariant vector bundle. The corresponding coadjoint orbit in $\mathfrak{g}_{\mathbb{R}}^*$ under the KSV correspondence is exactly the orbit associated to the corresponding $G_{\mathbb{R}}$-representation $V$ under the Mackey analogy via the usual orbit method. Moreover, I plan to show that the KSV correspondence and the Mackey correspondence fit nicely with the Matsuki correspondence ([49], [51], [55]).

Even in the case of $SL(2,\mathbb{R})$, this new perspective provides a solution to the unsettled debate in orbit method over what representations should be assigned to the nilpotent orbits. For instance, from the KSV correspondence it seems to natural to assign the two limit of discrete representations to the two half cones and the two spherical principal series representations to the entire nilpotent orbits.

In [33] Gukov and Witten proposed to use hyperkähler structures on complex coadjoint orbits and string theory to quantization branes to get unitary representations. In particular, they suggest a way to obatin complementary representations of $SL(2,\mathbb{R})$. I hope that my approach will make their construction more precise.

## 4. Oka principle and Connes-Kasparov

Let $G$ be a reductive Lie group. The representation theory of $G$ is closely related to various convolution algebras of functions on the group. From the perspective of noncommutative geometry, the natural algebra to study is the reduced group $C^*$-algebra $C^*_r(G)$, which captures the tempered dual of $G$. The well-known Connes-Kasparov Conjecture calculates the $K$-theory of $C^*_r(G)$ in terms of its maximal compact subgroup $K$ ([6]).

Together with Jonathan Block and Nigel Higson, we want to give a new proof of Connes-Kasparov from a complex geometric perspective. We consider a second convolution algebra, Harish-Chandra's Schwartz algebra $\mathcal{HC}(G)$ of smooth functions which decay faster than any polynomial weight at infinity (and the same for all derivatives). $\mathcal{HC}(G)$ captures the tempered dual of $G$ and, by a deep and difficult theorem of Arthur ([3]), it is isomorphic under a Fourier-type transform to the algebra of Schwartz sections on the tempered dual with valued in the bundle of tempered unitary irreducible representations. However, the result requires hard harmonic analysis on the group.

Hence we consider the third convolution algebra, Casselman's Schwartz algebra $\mathcal{S}(G)$ ([24]) of smooth functions on $G$ which decay faster than any exponential weight at infinity (and the same for all derivatives). The important Casselman-Wallach globalization theorem essentially asserts that any Harish-Chandra module of $G$ can be embedded into a smooth admissible moderate growth Fréchet representation ($SF$-representation) of $G$, which is unique up to isomorphism. An $SF$-representation is equvalently a nondegenerate continuous $\mathcal{S}(G)$-module. Hence $\mathcal{S}(G)$-modules essentially captures the admissible dual of $G$. Analogous to the theorem of Arthur and the usual Paley-Wiener theorem, $\mathcal{S}(G)$ can be viewed hypothetically as holomorphic sections with Schwartz-type decay of some bundle over the admissible dual. Following the philosophy that admissible dual is some kind of complexification of the tempered dual and the tempered dual is a retraction of the admissible dual, we propose to consider the algebra $A_{hol} = \mathcal{S}(G) \otimes_{\mathcal{Z}\mathfrak{g}} \mathcal{O}(\mathfrak{h})^W$, where $\mathcal{Z}\mathfrak{g}$ is the center of the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ and acts on $\mathcal{S}(G)$ as bi-invariant differential operators, which by Harish-Chandra isomorphism is the same as the algebra of polynomial functions on the dual of a Cartan algebra $\mathfrak{h}$ invariant under the action of the Weyl group $W$, and $\mathcal{O}(\mathfrak{h}^*)^W$ is the $W$-invariant of the algebra of holomorphic functions over $\mathfrak{h}^*$ with certain Schwartz decay.

Similarly let $C(\mathfrak{h}^*)^W$ be the algebra of smooth Schwartz functions over $\mathfrak{h}$ with $W$-symmetry and form $A = \mathcal{S}(G) \otimes_{\mathcal{Z}\mathfrak{g}} C(\mathfrak{h}^*)^W$. Both $A_{hol}$ and $A$ are Fréchet algebras.

Inspired by the *Oka principle* ([52]) in classical comlex geometry, which essentially states that the category of smooth vector bundles over a Stein complex manifold is equivalent to the category of holomorphic vector bundles over the same manifold, and a noncommutative version by Bost ([17]), we make the following conjecture.

**Conjecture 4.1.** *The natural homomorphism $A_{hol} \to A$ induces an isomorphism between the topological $K_0$-groups,*

$$K_0(\mathcal{S}(G) \otimes_{\mathcal{Z}\mathfrak{g}} \mathcal{O}(\mathfrak{h}^*)^W) \xrightarrow{\sim} K_0(\mathcal{S}(G) \otimes_{\mathcal{Z}\mathfrak{g}} C(\mathfrak{h}^*)^W). \tag{4.1}$$

We also conjecture that $K_0(A) \simeq K_0(\mathcal{HC}(G))$. Since $K_0(\mathcal{HC}(G)) \simeq K_0(C_r^*(G))$, the right hand of the isomorphism (4.1) is the same as $K_0(C_r^*(G)$. $K_0(\mathcal{S}(G) \otimes_{\mathcal{Z}\mathfrak{g}} \mathcal{O}(\mathfrak{h})^W)$ is easier to compute and the isomorphism (4.1) is essentially equivalent to the Connes-Kasparov isomorphism.

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