1) Suppose S is the portion of the plane $x+2y+2z=6$ that lies inside the first octant given by the mapping:

$$\Phi(x,y) = \left(x, y, \frac{6-x-2y}{2}\right)$$

and

$$\omega = xdy \wedge dz + ydz \wedge dx + 3zdx \wedge dy$$

is a 2-form over $\mathbb{R}^3$. Compute the integral of ω over S.

Note that

$$\int_S \omega = \int_D 3z\frac{\partial(x,y)}{\partial(x,y)} + x\frac{\partial(y,z)}{\partial(x,y)} + y\frac{\partial(z,x)}{\partial(x,y)}dxdy$$

Calculating the Jacobians, we get:

$$< \frac{\partial(x,y)}{\partial(x,y)}, \frac{\partial(y,z)}{\partial(x,y)}, \frac{\partial(z,x)}{\partial(x,y)} >=< 1, \frac{1}{2}, 1 >$$

Substituting:

$$\begin{aligned}\int_S \omega &= \int_D 3(\frac{6-x-2y}{2}) + \frac{1}{2}x + y\, dxdy \\ &= \int_0^3 \int_0^{6-2y} 3(\frac{6-x-2y}{2}) + \frac{1}{2}x + y\, dxdy \\ &= \int_0^3 \int_0^{6-2y} 9 - x - 2y + y\, dxdy \\ &= \int_0^3 36 - 18y + 2y^2\, dy \\ &= \boxed{45}\end{aligned}$$

2) Let S be the graph of $z = x^2+y^2+7$ over the disk $x^2+y^2 \leq 9$. Let $\omega = x^2(dx \wedge dy)$. Calculate the integral of ω over S.

Parametrize the surface S using the map $\Phi(s,t) = (s,t,s^2+t^2+7)$. Then $\frac{\partial(x,y)}{\partial(s,t)} = 1$ and we get:

$$\begin{aligned}\int_S \omega &= \int_D (s^2+t^2+7)^2\frac{\partial(x,y)}{\partial(s,t)}dsdt \\ &= \int_D (s^2+t^2+7)^2 dsdt\end{aligned}$$

It will be easier to integrate over D in polar coordinates, so let $s = r\cos(\theta)$ and $t = r\sin(\theta)$. The Jacobian resulting from changing to polar coordinates is r, so:

$$\begin{aligned}\int_D (s^2+t^2+7)^2 ds dt &= \int_0^{2\pi}\int_0^3 r(r^2+7)^2 dr d\theta \\ &= \int_0^{2\pi} \frac{1251}{2} d\theta \\ &= \boxed{1251\pi}\end{aligned}$$

3) Let

$$\omega = z dy \wedge dz + (y+1) dz \wedge dx - dx \wedge dy$$

and let

$$T(u,v) = (u^2+v, u-v, uv)$$

be a mapping from $\mathbb{R}^2 \to \mathbb{R}^3$. Find the pullback $T^*\omega$

Note that:

$$\begin{aligned} dx &= \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv = (2u) du + 1 dv \\ dy &= \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv = 1 du - 1 dv \\ dz &= \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = v du + u dv \end{aligned}$$

So:

$$\begin{aligned} T^*\omega = &\ uv((du-dv)\wedge(vdu+udv)) + (u-v+1)((vdu+udv)\wedge(2udu+1dv)) \\ &-(2u(du\wedge dv)\wedge(du-dv)) \\ = &\ u^2v(du\wedge dv) - uv^2(dv\wedge du) + v(u-v+1)(du\wedge dv) \\ &+2u^2(u-v+1)(dv\wedge du) + 2u(du\wedge dv) - 1(dv\wedge du) \\ = &\ \boxed{(-2u^3+3u^2v-2u^2+uv^2+uv+2u-v^2+v+1)(du\wedge dv)} \end{aligned}$$

4) Let

$$\omega = 2xy dx + (x^2-z^2) dy - 2yz dz$$

(a) Check that ω is a closed form

(b) Determine whether there exists a smooth function $f(x, y, z)$ over $\mathbb{R}^3$ such that $df = \omega$. Find all f satisfying this condition.

(c) Find the line integral of ω along the parametrized curve $\gamma(t) = (1-t, \sin(\pi t), t^2+2)$ for $0 \leq t \leq 1$.

(a) To check that ω is closed, we need to show that $d(\omega) = 0$. So:

$$\begin{aligned} d(\omega) &= d(2xydx + (x^2 - z^2)dy - 2yzdz) \\ &= (2ydx + 2xdy) \wedge dx + (2xdx - 2zdz) \wedge dy - (2zdy + 2ydz) \wedge dz \\ &= 2x(dy \wedge dx) + 2x(dx \wedge dy) - 2z(dz \wedge dy) - 2z(dy \wedge dz) \\ &= 0 \end{aligned}$$

(b) We need:

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \omega \end{aligned}$$

This implies that:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2xy \\ \frac{\partial f}{\partial y} &= x^2 - z^2 \\ \frac{\partial f}{\partial z} &= -2yz \end{aligned}$$

An f that satisfies such conditions is $\boxed{f = x^2y - yz^2 + c}$, where c is some constant.

(c) Using Stokes's theorem, which states that:

$$\int_{\partial\gamma} \phi = \int_{\gamma} d(\phi)$$

Noting that the boundary of a line is described by its endpoints

$$\begin{aligned} \int_{\gamma} d(\phi) &= \phi(< 0, 0, 3 >) - \phi(< 1, 0, 2 >) \\ &= \boxed{0} \end{aligned}$$

5) Let

$$\omega = (12x^2y^3 + 2y)(dx \wedge dy)$$

Check that ω is closed and find a smooth ϕ s.t. $d\omega = \phi$.

$$\begin{aligned} d(\omega) &= d(12x^2y^3 + 2y) \wedge dx \wedge dy \\ &= ((24xy^3)dx + (36x^2y^2 + 2)dy) \wedge dx \wedge dy \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

To find a ϕ such that $d(\phi) = \omega$, let $\phi = Adx + Bdy$. Then:

$$\begin{aligned} d(\phi) &= (\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy) \wedge (dx) + (\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy) \wedge dy \\ &= (\frac{\partial A}{\partial y})(dy \wedge dx) + (\frac{\partial B}{\partial x})(dx \wedge dy) \end{aligned}$$

Thus, we can set $A = 0$ and find a B such that $\frac{\partial B}{\partial x} = 12x^2y^3 + 2y$. If we simply integrate $12x^2y^3 + 2y$ w.r.t. x, we get:

$$B = 4x^3y^3 + 2xy$$

This implies that a ϕ such that $d(\phi) = \omega$ is:

$$\phi = (4x^3y^3 + 2xy)\, dy$$

6) Let

$$\omega = (3y - 2yz)(dx \wedge dy) + y^2(dz \wedge dx) + 2z(dy \wedge dz)$$

Check that ω is closed and find a smooth ϕ s.t. $d\omega = \phi$.

To check that ω is closed:

$$\begin{aligned} d\omega &= (3dy - 2ydz) \wedge (dx \wedge dy) + 2ydy \wedge dz \wedge dx + 2dz \wedge dy \wedge dz \\ &= -2y(dz \wedge dx \wedge dy) + 2y(dy \wedge dz \wedge dz) \\ &= 0 \end{aligned}$$

Now, let $\phi = Pdx + Qdy + Rdz$. Then:

$$\begin{aligned} d\phi = (&\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz) \wedge dx \\ &+(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz) \wedge dy \\ &+(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \frac{\partial R}{\partial z}dz) \wedge dz \end{aligned}$$

$$= (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})(dx \wedge dy) + (\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z})(dx \wedge dz) + (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})(dy \wedge dz)$$

Thus, a general form of the solution will satisfy the following relationships:

$$\begin{aligned} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= 3y - 2yz \\ \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} &= -y^2 \\ \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} &= 2z \end{aligned}$$

To simplify, force $R = 0$. Then:

$$\begin{aligned} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= 3y - 2yz \\ \frac{\partial P}{\partial z} &= y^2 \\ \frac{\partial Q}{\partial z} &= -2z \end{aligned}$$

Let $P = zy^2$ and $Q = 3yx - z^2$. Then:

$$\begin{aligned} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= 3y - 2yz \\ \frac{\partial P}{\partial z} &= y^2 \\ \frac{\partial Q}{\partial z} &= -2z \end{aligned}$$

as we wanted. Thus, one such ϕ s.t. $d\phi = \omega$ is:

$$\boxed{\phi = zy^2dx + (3yx - z^2)dy}$$