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Math 361: Homework 2 Solutions

1. Let U be an open subset in $\mathbb{R}^n$, $f, g : U \to \mathbb{R}^m$ be two differentiable functions and a, b be any two real numbers. Show that $af + bg$ is again differentiable and

$$D(af + bg) = aDf + bDg$$

Since f is differentiable, we know that for all $x \in U$, we can write:

$$f(x+h) - f(x) = f'(x)h + r_f(h) \tag{1}$$

where $f'(x)$ is Df evaluated at x and where $\lim_{h\to 0} \frac{|r_f(h)|}{|h|} = 0$.

Similarly, since g is differentiable, we can do the same for a sublinear remainder $r_g(h)$:

$$g(x+h) - g(x) = g'(x)h + r_g(h) \tag{2}$$

Next, we try to find the derivative of $af + bg$ by considering $(af + bg)(x + h) - (af + bg)(x)$:

$$\begin{aligned}
(af + bg)(x+h) - (af + bg)(x) &= af(x+h) + bg(x+h) - (af(x) + bg(x)) && (3)\\
&= a(f(x+h) - f(x)) + b(g(x+h) - g(x)) && (4)\\
&= a(f'(x)h + r_f(h)) + b(g'(x)h + r_g(h)) && (5)\\
&= (af'(x) + bg'(x))h + ar_f(h) + br_g(h) && (6)
\end{aligned}$$

Now, we argue that $(ar_f(h) + br_g(h))$ is sublinear:

$$\begin{aligned}
\lim_{h\to 0} \frac{|ar_f(h) + br_g(h)|}{|h|} &\leq \lim_{h\to 0} \frac{|ar_f(h)| + |br_g(h)|}{|h|} && \text{(triangle inequality)} \quad (7)\\
&= \lim_{h\to 0} \frac{|ar_f(h)|}{|h|} + \lim_{h\to 0} \frac{|br_g(h)|}{|h|} && \text{(limits exist individually)} \quad (8)\\
&= |a| \lim_{h\to 0} \frac{|r_f(h)|}{|h|} + |b| \lim_{h\to 0} \frac{|br_g(h)|}{|h|} && (a, b \in \mathbb{R}) \quad (9)\\
&= |a| \cdot 0 + |b| \cdot 0 = 0 && \text{(differentiability of } f, g) \quad (10)
\end{aligned}$$

We have shown that $\lim_{h\to 0} \frac{|ar_f(h) + br_g(h)|}{|h|} \leq 0$, but because norms are always non-negative, this indicates that the limit is actually equal to 0.

We let $ar_f(h) + br_g(h) = r_{af+bg}(h)$. We have shown that:

$$(af + bg)(x+h) - (af + bg)(x) = (af'(x) + bg'(x))h + r_{af+bg}(h) \tag{11}$$

where $\frac{|r_{af+bg}(h)|}{|h|} \to 0$ as $h \to 0$. Therefore, by the Landau definition of differentiability, we have shown that $af + bg$ is differentiable at every point $x \in U$ and that its derivative is equal to $af'(x) + bg'(x) = aDf + bDg$. Note that this derivative is unique by Theorem 9.12 in Rudin.

2. Let U be an open subset in $\mathbb{R}^n$, and $f, g : U \to \mathbb{R}^m$ be two differentiable real-valued functions. Show that the product function fg is again differentiable and

$$D(fg) = gDf + fDg$$

Note that here g times Df as a $1 \times n$ matrix is just the entrywise multiplication.

We set up the problem in the same way that we did problem 1: Since f, g are both differentiable for all $x \in U$:

$$f(x+h) - f(x) = f'(x)h + r_f(h) \tag{1}$$

$$g(x+h) - g(x) = g'(x)h + r_g(h) \tag{2}$$

where $r_f(h)$ and $r_g(h)$ are sublinear.

Next, we try to find the derivative of fg by considering $(fg)(x+h) - (fg)(x)$:

$$\begin{aligned}
(fg)(x+h) - (fg)(x) &= f(x+h)g(x+h) - f(x)g(x) && (3)\\
&= f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x) && (4)\\
&= f(x+h)(g(x+h) - g(x)) + g(x)(f(x+h) - f(x)) && (5)\\
&= f(x+h)(g'(x)h + r_g(h)) + g(x)(f'(x)h + r_f(h)) && (6)\\
&= f(x+h)g'(x)h + f(x+h)r_g(h) + g(x)f'(x)h + g(x)r_f(h) && (7)\\
&= [f(x) + f'(x)h + r_f(h)]g'(x)h + f(x+h)r_g(h) + g(x)f'(x)h + g(x)r_f(h) && (8)\\
&= [f(x)g'(x) + g(x)f'(x)]h + [f(x+h)r_g(h) + g(x)r_f(h) + f'(x)g'(x)h^2 && (9)\\
&\quad + r_f(h)g'(x)h]
\end{aligned}$$

Note that in the above calculation we subbed in for $g(x+h) - g(x)$, $f(x+h) - f(x)$ and then again for just $f(x+h)$ all from the definition of differentiability in (1) and (2).

We let $r_{fg} = f(x+h)r_g(h) + g(x)r_f(h) + f'(x)g'(x)h^2 + r_f(h)g'(x)h$. Now, we argue that each term in r_{fg} is actually sublinear:

$$\lim_{h\to 0} \frac{|f(x+h)r_g(h)|}{|h|} = \underbrace{\lim_{h\to 0} |f(x+h)|}_{\text{exists because } f \text{ continuous}} \cdot \underbrace{\lim_{h\to 0} \frac{|r_g(h)|}{|h|}}_{\text{exists because } g \text{ differentiable}} = |f(x)| \cdot 0 = 0 \tag{10}$$

$$\tag{11}$$

$$\lim_{h\to 0} \frac{|g(x)r_f(h)|}{|h|} = \underbrace{|g(x)|}_{\text{doesn't depend on } h} \cdot \lim_{h\to 0} \frac{|r_f(h)|}{|h|} = |g(x)| \cdot 0 = 0 \tag{12}$$

$$\tag{13}$$

$$\lim_{h\to 0} \frac{|f'(x)g'(x)h^2|}{|h|} = \underbrace{|f'(x)g'(x)|}_{\text{doesn't depend on } h} \cdot \lim_{h\to 0} \frac{|h^2|}{|h|} = |f'(x)g'(x)| \cdot \lim_{h\to 0} |h| = |f'(x)g'(x)| \cdot 0 = 0 \tag{14}$$

$$\tag{15}$$

$$\lim_{h\to 0} \frac{|r_f(h)g'(x)h|}{|h|} = \underbrace{|g'(x)|}_{\text{doesn't depend on } h} \cdot \underbrace{\lim_{h\to 0} \frac{|r_f(h)|}{|h|}}_{\text{exists because } f \text{ differentiable}} \cdot \lim_{h\to 0} |h| = |g'(x)| \cdot 0 \cdot 0 = 0 \tag{16}$$

Now, since each term is sublinear, it must mean that r_{fg} is sublinear:

$$\begin{aligned}
\lim_{h\to 0} \frac{|r_{fg}|}{|h|} &\le \lim_{h\to 0} \frac{|f(x+h)r_g(h)|}{|h|} + \lim_{h\to 0} \frac{|g(x)r_f(h)|}{|h|} + \lim_{h\to 0} \frac{|f'(x)g'(x)h^2|}{|h|} + \lim_{h\to 0} \frac{|r_f(h)g'(x)h|}{|h|} && (17)\\
&= 0 + 0 + 0 + 0 = 0 && (18)
\end{aligned}$$

This implies that $\lim_{h\to 0} \frac{|r_{fg}|}{|h|}$ actually equals zero because the norms can never be negative.

Therefore, we have shown that:

$$(fg)(x+h) - (fg)(x) = (f(x)g'(x) + g(x)f'(x))h + r_{fg}(h) \tag{19}$$

where $\frac{|r_{fg}(h)|}{|h|} \to 0$ as $h \to 0$. Therefore, by the Landau definition of differentiability, we have shown that fg is differentiable at every point $x \in U$ and that its derivative is equal to $f(x)g'(x)+g(x)f'(x) = fDg + gDf$. Note that this derivative is unique by Theorem 9.12 in Rudin.

3. Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Show that $T : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable as a map and compute its total derivative.

In this question, we will use the limit definition of differentiability (just to mix it up a little). A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^m$ if there exists some linear transformation A of $\mathbb{R}^n$ into $\mathbb{R}^m$ such that:

$$\lim_{h\to 0} \frac{|f(x+h) - f(x) - A(h)|}{|h|} = 0 \tag{1}$$

If this is the case, we call A the total derivative of f.

Now, let us plug in our function T into this definition. We claim that T is differentiable and that its derivative is just T itself (i.e., $A = T$).

$$\begin{aligned}
\lim_{h\to 0} \frac{|T(x+h) - T(x) - A(h)|}{|h|} &= \lim_{h\to 0} \frac{|T(x) + T(h) - T(x) - A(h)|}{|h|} && (T \text{ is linear}) \quad (2)\\
&= \lim_{h\to 0} \frac{|T(h) - A(h)|}{|h|} && (T(x) \text{ cancels}) \quad (3)\\
&= \lim_{h\to 0} \frac{|T(h) - T(h)|}{|h|} && (\text{we get to choose } A) \quad (4)\\
&= \lim_{h\to 0} \frac{0}{|h|} = 0 && (5)
\end{aligned}$$

Therefore, we have found that there does exist a transformation that makes the limit in (1) go to 0, which means that T is indeed differentiable and that its total derivative is itself. In addition, we know that this is the unique total derivative by the same Theorem 9.12 in Rudin as before.

4. Pugh Chapter 5 #20: Assume that U is a connected open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is differentiable everywhere on U. If $(Df)_p = 0$ for all $p \in U$, show that f is constant.

Lemma: f is locally constant, i.e., for any $x \in U$, there always exists an open neighborhood $N_\delta(x)$ of x inside U such that f is constant over $N_\delta(x)$.

Since f is continuous and defined on an open domain, we can take δ small enough to form an open ball $N_\delta(x)$ for any $x \in U$ such that $N_\delta(x) \subset U$. Now, we want to show that f must be constant under all elements in $N_\delta(x)$.

Let p be an arbitrary point inside $N_\delta(x)$. We know that the segment $[x, p]$ is contained within U because $x, p \in N_\delta(x) \subset U$. Since f is differentiable, by the multivariable Mean Value Theorem, we know that:

$$|f(p) - f(x)| \leq M|p - x| \text{ where } M = \sup_{x\in U} \|(Df)_x\| \tag{1}$$

However, we know that $(Df)_x = 0$ for all $x \in U$. This means that $M = \sup_{x\in U} \|0\| = 0$. Hence, we have that:

$$|f(p) - f(x)| \leq 0 \cdot |p - x| \implies |f(p) - f(x)| \leq 0 \implies f(p) - f(x) = 0 \implies f(p) = f(x) \tag{2}$$

In other words, we have just shown that for any $p \in N_\delta(x)$, it must be the case that $f(p) = f(x)$, i.e., the function f is locally constant. Therefore, we have proven the lemma and we now move on to proving that f must be constant over all of U.

We argue by contradiction. Suppose f is not constant. Then we know that the cardinality of $\text{Im}(f)$ must be strictly greater than 1. Let $y \in \text{Im}(f)$, and consider the following two sets:

$$A = f^{-1}(\{y\}) \qquad\qquad B = f^{-1}(\text{Im}(f) \setminus \{y\}) \tag{3}$$

We argue that the sets A and B have the following characteristics:

- A and B are both nonempty.

  We know that A is nonempty because $y \in \text{Im}(f)$, which means there must exist at least one element in U that maps to $\{y\} \implies f^{-1}(\{y\})$ is nonempty.

  Since the cardinality of $\text{Im}(f)$ is greater than 1, it must be the case that $\text{Im}(f) \setminus \{y\}$ is nonempty. Therefore, similarly to above, there must be at least one element in U that maps to $\text{Im}(f) \setminus \{y\}$ which implies that $f^{-1}(\text{Im}(f) \setminus \{y\})$ is nonempty.

- $A \cup B = U$

  It's clear that $\{y\} \cup \text{Im}(f) \setminus \{y\} = \text{Im}(f)$, and by definition we know that $f^{-1}(\text{Im}(f)) = U$. Therefore, it follows that $f^{-1}(\{y\} \cup \text{Im}(f) \setminus \{y\}) = U$. Finally, since $\{y\}$ and $\text{Im}(f) \setminus \{y\}$ are disjoint, the previous statement must imply that the union of their inverse images must make up all of U, i.e., $f^{-1}(\{y\}) \cup f^{-1}(\text{Im}(f) \setminus \{y\}) = U$. That is, $A \cup B = U$.

- $A \cap B = \emptyset$

  Suppose not. This must mean that there is an element $x \in U$ such that $f(x) = y$ and $f(x) = z$ where $z \in \text{Im}(f) \setminus \{y\}$. This means that $f(x)$ equals two distinct elements in $\mathbb{R}^n$, which would mean that f is not a function. Therefore, we have a contradiction and we have shown that $A \cap B = \emptyset$.

- A and B are both open.

  Let x be any arbitrary element in A, i.e., $f(x) = y$. Since $A \subset U$, by our lemma we know that we can form a $\delta-$neighborhood around x (call it $N_\delta(x)$) such that the value of f under $N_\delta(x)$ is constant. That is, for any $t \in N_\delta(x)$, we know that $f(t) = y$. In other words, $f(N_\delta(x)) = \{y\}$ which implies that $N_\delta(x) \subset f^{-1}(\{y\})$. Therefore, we have shown that for any x in A, $N_\delta(x) \subset A$, which means that A must be open.

  The exact same argument works for B, since our lemma holds for any element in U: Let x be any arbitrary element in B. We form a $\delta-$neighborhood around x (call it $N_\delta(x)$) such that the value of f under $N_\delta(x)$ is constant. That is, for any $t \in N_\delta(x)$, we know that $f(t) = f(x) \in \text{Im}(f) \setminus \{y\}$. In other words, $f(N_\delta(x)) = \text{Im}(f) \setminus \{y\}$ which implies that $N_\delta(x) \subset f^{-1}(\text{Im}(f) \setminus \{y\})$. Therefore, we have shown that for any x in B, $N_\delta(x) \subset B$, which means that B must be open.

Hence, we have shown that there exist two open, nonempty, disjoint sets A and B whose union is U. This must mean that U is disconnected, which contradicts our assumption of U being connected. Therefore, we know that the cardinality of $\text{Im}(f)$ must equal 1, i.e., f is constant.

7. Pugh Chapter 5 #24: Show that all second partial derivatives of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

exist everywhere, but the mixed second partials are unequal at the origin, $\partial^2 f(0,0)/\partial x\partial y \neq \partial^2 f(0,0)/\partial y\partial x$.

To show that the partial derivatives exist everywhere, we take the derivatives with respect to x and y by holding the other variable constant and using the product rule and chain rule. For $(x,y) \neq (0,0)$:

$$\frac{\partial f}{\partial x} = xy(x^2-y^2)\cdot\frac{-2x}{(x^2+y^2)^2} + (xy\cdot 2x + y(x^2-y^2))\frac{1}{x^2+y^2} \tag{1}$$

$$\implies \frac{\partial f}{\partial x} = \begin{cases} \dfrac{-2x^2y(x^2-y^2)}{(x^2+y^2)^2} + \dfrac{2x^2y + y(x^2-y^2)}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases} \tag{2}$$

We calculate the partial with respect to y similarly:

$$\frac{\partial f}{\partial y} = xy(x^2-y^2)\cdot\frac{-2y}{(x^2+y^2)^2} + (xy\cdot(-2y) + x(x^2-y^2))\frac{1}{x^2+y^2} \tag{3}$$

$$\implies \frac{\partial f}{\partial x} = \begin{cases} \dfrac{-2xy^2(x^2-y^2)}{(x^2+y^2)^2} + \dfrac{-2xy^2 + x(x^2-y^2)}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases} \tag{4}$$

In order to show that all second order partial derivatives exist, we would have to take the derivative of $\frac{\partial f}{\partial x}$ twice more (once with respect to x and again with respect to y) and the derivative of $\frac{\partial f}{\partial y}$ twice more as well. However, notice that when we take the derivatives, we always have some polynomial of x and y in the numerator and a power of (x^2+y^2) in the denominator. This indicates that the partials will exist whenever the denominator is nonzero (because polynomials are continuous), and the denominator is zero only at $(0,0)$, which we define to equal zero. This means that all of the partial derivatives will exist everywhere.

To show that that the mixed partials do not agree at the origin, we calculate the mixed second partial derivatives of f using the limit definition of partial derivatives at a point:

$$\frac{\partial^2 f}{\partial y\partial x}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(0,0)\right) = \frac{\partial}{\partial y}\left(\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}\right) = \tag{5}$$

$$= \lim_{k\to 0}\left(\frac{\lim_{h\to 0}\dfrac{f(0+h,0+k)-f(0,0+k)}{h} - \lim_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h}}{k}\right) = \tag{6}$$

$$= \lim_{k\to 0}\left(\frac{\lim_{h\to 0}\dfrac{\dfrac{hk(h^2-k^2)}{h^2+k^2} - \dfrac{0\cdot k(0-k^2)}{0+k^2}}{h} - \lim_{h\to 0}\dfrac{\dfrac{h\cdot 0(h^2-0)}{h^2+0} - 0}{h}}{k}\right) = \tag{7}$$

$$= \lim_{k\to 0}\left(\frac{\lim_{h\to 0}\dfrac{hk(h^2-k^2)}{h(h^2+k^2)} - \lim_{h\to 0}\dfrac{0-0}{h}}{k}\right) = \lim_{k\to 0}\left(\frac{\dfrac{k(0-k^2)}{(0+k^2)} - 0}{k}\right) = \tag{8}$$

$$= \lim_{k\to 0}\left(\frac{-k^3}{k^3}\right) = \lim_{k\to 0} -1 = -1 \tag{9}$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(0,0)\right) = \frac{\partial}{\partial x}\left(\lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h}\right) = \tag{10}$$

$$= \lim_{k\to 0}\left(\frac{\lim_{h\to 0}\frac{f(0+k,0+h)-f(0+k,0)}{h} - \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h}}{k}\right) = \tag{11}$$

$$= \lim_{k\to 0}\left(\frac{\lim_{h\to 0}\frac{\frac{kh(k^2-h^2)}{k^2+h^2} - \frac{k\cdot 0(k^2-0)}{k^2+0}}{h} - \lim_{h\to 0}\frac{\frac{0\cdot h(0-h^2)}{0+h^2} - 0}{h}}{k}\right) = \tag{12}$$

$$= \lim_{k\to 0}\left(\frac{\lim_{h\to 0}\frac{kh(k^2-h^2)}{h(k^2+h^2)} - \lim_{h\to 0}\frac{0-0}{h}}{k}\right) = \lim_{k\to 0}\left(\frac{\frac{k(k^2-0)}{(k^2+0)} - 0}{k}\right) = \tag{13}$$

$$= \lim_{k\to 0}\left(\frac{k^3}{k^3}\right) = \lim_{k\to 0} 1 = 1 \tag{14}$$

Therefore, we have shown that:

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1 \neq -1 = \frac{\partial^2 f}{\partial y \partial x}(0,0) \tag{15}$$

8. Rudin Chapter 9 #28: For $t \geq 0$, put

$$\varphi(x,t) = \begin{cases} x & (0 \leq x \leq \sqrt{t}) \\ -x + 2\sqrt{t} & (\sqrt{t} \leq x \leq 2\sqrt{t}) \\ 0 & \text{(otherwise)}, \end{cases}$$

and put $\varphi(x,t) = -\varphi(x,|t|)$ if $t < 0$. Show that φ is continuous on $\mathbb{R}^2$, and

$$(D_2\varphi)(x,0) = 0$$

for all x. Define

$$f(t) = \int_{-1}^{1} \varphi(x,t)dx$$

Show that $f(t) = t$ if $|t| < \frac{1}{4}$. Hence

$$f'(0) \neq \int_{-1}^{1} (D_2\varphi)(x,0)dx.$$

To show that φ is continuous on $\mathbb{R}^2$, we first note that the three functions that make up φ (x, $-x+2\sqrt{t}$, and 0) are all continuous. Therefore, we just have to show that the pieces of φ agree on the boundaries, i.e., at $x = 0$, at $x = \sqrt{t}$, and at $x = 2\sqrt{t}$.

$$\lim_{x\to 0^-} \varphi(x,t) = \lim_{x\to 0} 0 = 0 \tag{1}$$

$$\lim_{x\to 0^+} \varphi(x,t) = \lim_{x\to 0} x = 0 \tag{2}$$

Therefore, the left limit and right limit agree and φ is continuous at $x = 0$.

$$\lim_{x \to \sqrt{t}^-} \varphi(x,t) = \lim_{x \to \sqrt{t}} x = \sqrt{t} \tag{3}$$

$$\lim_{x \to \sqrt{t}^+} \varphi(x,t) = \lim_{x \to \sqrt{t}} (-x + 2\sqrt{t}) = -\sqrt{t} + 2\sqrt{t} = \sqrt{t} \tag{4}$$

Therefore, φ is continuous at $x = \sqrt{t}$.

$$\lim_{x \to 2\sqrt{t}^-} \varphi(x,t) = \lim_{x \to 2\sqrt{t}} (-x + 2\sqrt{t}) = -2\sqrt{t} + 2\sqrt{t} = 0 \tag{5}$$

$$\lim_{x \to 2\sqrt{t}^+} \varphi(x,t) = \lim_{x \to 2\sqrt{t}} 0 = 0 \tag{6}$$

Therefore, φ is continuous at $x = 2\sqrt{t}$. That is, φ is continuous on every boundary and also everywhere else (because the functions that make it up are continuous), which means that φ is continuous on $\mathbb{R}^2$.

Next, we want to show that $(D_2\varphi)(x,0) = 0$. We do this by using the definition of partial derivatives:

$$(D_2\varphi)(x,0) = \lim_{h \to 0} \frac{\varphi(x, 0+h) - \varphi(x,0)}{h} = \lim_{h \to 0} \frac{x - x}{h} = \lim_{h \to 0} 0 = 0 \tag{7}$$

Note that we used the definition of φ close to 0 because as $h \to 0$ so does $\sqrt{h}$.

As for the last part of the problem, first consider $0 < t < \frac{1}{4}$. Note that in this case, $0 < \sqrt{t} < \frac{1}{4} \implies 0 < 2\sqrt{t} < 1$. This means that when we compute $f(t)$ for $0 < t < \frac{1}{4}$, we only need to worry about integrating between 0 and 1:

$$f(t) = \int_0^1 \varphi(x,t)dx = \int_0^{\sqrt{t}} xdx + \int_{\sqrt{t}}^{2\sqrt{2}} (-x + 2\sqrt{t})dx = \tag{8}$$

$$= \frac{x^2}{2}\bigg|_0^{\sqrt{t}} + \left(2\sqrt{t}x - \frac{x^2}{2}\right)\bigg|_{\sqrt{t}}^{2\sqrt{t}} = \frac{t}{2} - 0 + 4t - \frac{4t}{2} - \left(2t - \frac{t}{2}\right) = \tag{9}$$

$$= \frac{t}{2} + 2t - 2t + \frac{t}{2} = t \tag{10}$$

When $t = 0$, $\varphi(x,t) = 0$ for any x. Therefore, we know that $f(0) = 0$. When $-\frac{1}{4} < t < 0$, we know that $\varphi(x,t) = -\varphi(x,|t|)$, which is to say that for these values of t:

$$f(x,t) = -f(x,-t) = -(-t) = t \tag{11}$$

Therefore, it follows that for any $|t| < \frac{1}{4}$, $f(t) = t$. By normal single-variable differentiation, this implies that $f'(t) = 1$ for these t values.

We have shown that $(D_2\varphi)(x,0) = 0$, therefore:

$$f'(0) = 1 \neq 0 = \int_{-1}^1 (D_2\varphi)(x,0)dx \tag{12}$$