Math 760

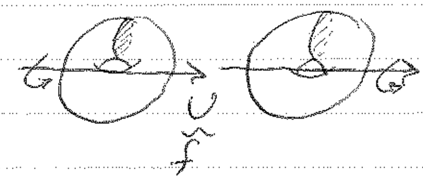
Def: A Heegaard splitting of a closed orientable manifold M is a triple (H_1, H_2, f) s.t. H_1 and H_2 are handle bodies, $f: \partial H_1 \to \partial H_2$ is a homeomorphism and $M \cong H_1 \amalg H_2 /_{\sim}$

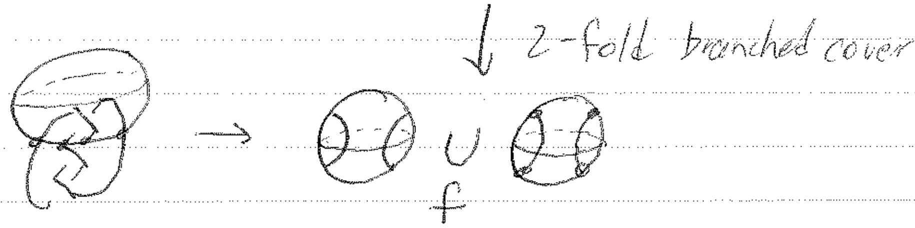
where $x \sim y$ if $x \in \partial H_1$ or $y \in \partial H_2$ and $f(x) = y$.

Def: A bridge splitting for (S^3, K) is n-strand a triple (T_1, T_2, f) s.t. T_1 and T_2 are trivial tangles, $f: \partial T_1 \to \partial T_2$ is a homeo. of the $2n$-punctured spheres ~~s.t.~~ and

$$(S^3, K) = T_1 \cup T_2 /_{\sim}$$

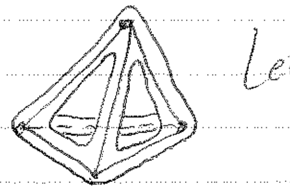
Ex| $L_{3/1} \to$





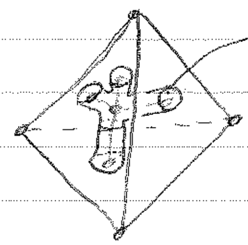
Th^m Every closed orientable ~~Heegaard splitting~~ 3-manifold has a Heegaard splitting.

Pf| By Moise, every 3-manifold has a triangulation $\mathcal{T}$. Let τ be a 3-simplex in $\mathcal{T}$.



Let $H_1 = \eta(\mathcal{T}^1)$

Let $(\mathcal{T}^1)^*$ be the dual 1-skeleton for $\mathcal{T}$.



Let $H_2 = \eta((\mathcal{T}^1)^*)$

$M = H_1 \cup H_2$

Hence M has a Heegaard splitting. $\square$

<u>Thm</u>| Let K be a knot in S^3. (S^3, K) has a bridge splitting.

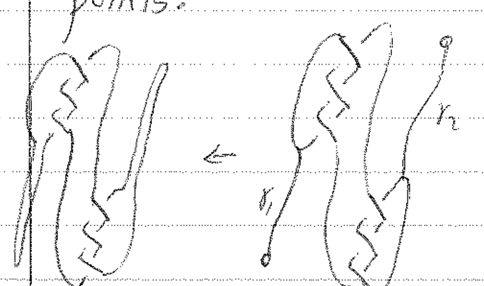
<u>Proof</u> Let $h : S^3 \to [-1,1]$ be the natural height function on S^3

$$S^3 \cong S^2 \times [-1,1]/\sim$$

$\sim$: Identify $S^2 \times \{-1\}$ to a point
& Identify $S^2 \times \{1\}$ to a point

$h : S^2 \times [-1,1]/\sim \ \to [-1,1]$
by $h(x,t) = t$.

By Morse theory we can assume $h|_K$ has finitly many, isolated ~~maxim~~ critical points.

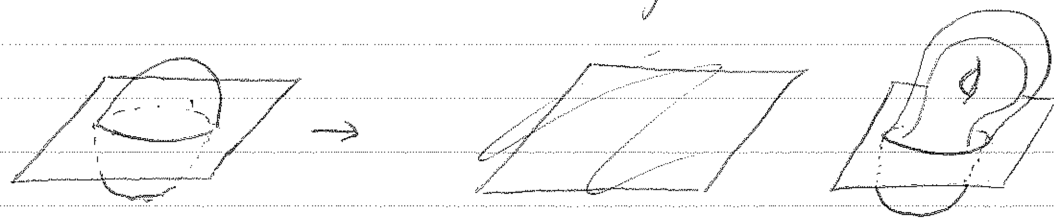


There is an isotopy of K supported in a nbh of γ_i s.t. after the isotopy all maxima of $h|_K$ lie above all minima.

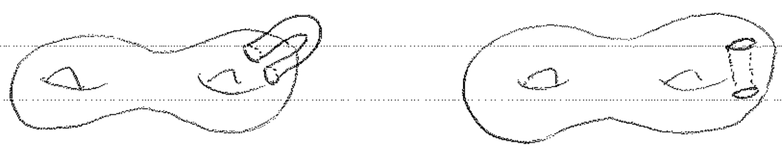
Any level sphere between the lowest max and highest min is a bridge sphere. $\square$

Are Bridge Heegaard splittings or Bridge splittings unique? No!

Stabelization of a Heegaard Surface

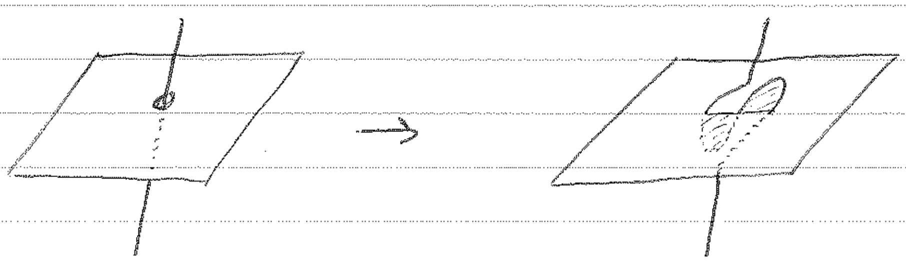


This has the effect of adding a handle to one handle body and drilling out an ~~unknot~~ boundary parallel arc from the other



Hence, every 3-manifold has ∞-many distinct Heegaard Splittings.

Stabelization of a Bridge Splittings



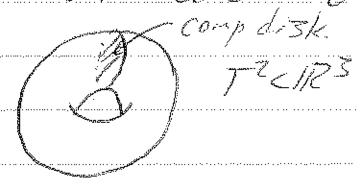
Ex) Justify that stabelizations of Bridge surfaces lift to stabelizations of Heegaard Splittings in 2-fold branched cover.

What do Heegaard Surfaces and Bridge Surfaces have in common?

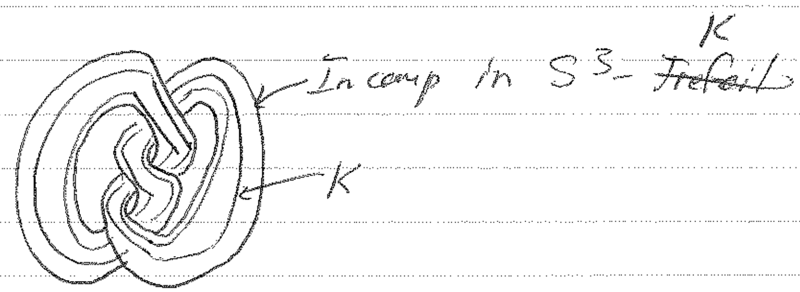
They are bicompressible!

Def:| A compressing disk for a surface $F \subset M$ is an embedded disk s.t. $D \cap F = \partial D$ and ∂D is essential in F.

Ex|



Def| If $F \subset M$ has no compressing disks, then F is incompressible.

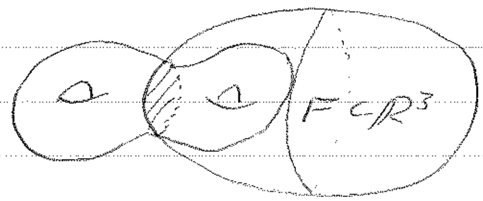


Def| If $F \subset M$ has a compressing disk to each side then F is bicompressible.

- Heegaard splitings
- Bridge Splittings.

How are Heegaard Splittings related to topology and geometry of the ambient 3-manifold?

Def: ~~A Heegaard spl.~~ A bicompressible surface $F \subset M$ is <u>reducible</u> if there exists an ess. curve in F that bounds comp. disks to both sides.



<u>Th^m</u> Any Heegaard ~~splitting~~ surface for a reducible 3-manifold is reducible.

<u>Pf</u> Next time.

Def: A bicomp. surface $F \subset M$ is weakly reducible if there exist disjoint essential curves γ_1 and γ_2 s.t. γ_1 bounds a comp disk to one side and γ_2 bounds a comp disk to the other side, and F is <u>not</u> reducible.

~~Def~~ Th^m (Casson & Gordon)
If M is a closed orientable 3-manifold with a weakly reducible Heegaard surface, then M contains an incompressible surface.