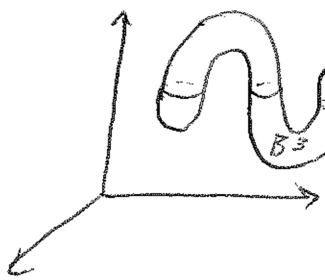
Math 760 Day 2

Taken from "Notes on Basic 3-Manifolds Topology"
By Allen Hatcher
Available free online

Preliminaries:

Alexanders Theorem: Every embedded 2-sphere in $\mathbb{R}^3$ bounds an embedded 3-ball.



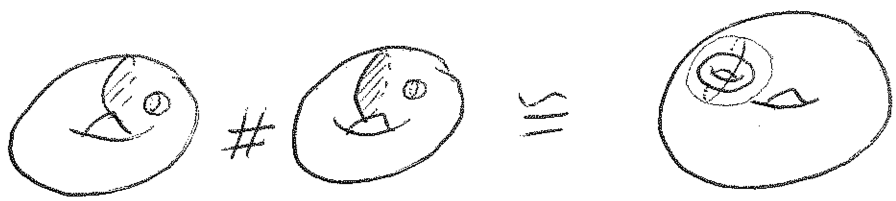
Def: A ~~compact conne~~ surface F in a 3-manifold M is 2-sided if there is an embedding $h: F \times I \to M$ s.t. $h(x, 1/2) = x$ for all $x \in F$ and $h(F \times I) \cap \partial M = h(\partial F \times I)$

Ex |

Thm | If F is a compact, orientable, properly embedded surface in a 3manifold M, then F is 2-sided.

Def If M is a connected 3-manifold and S is an embedded sphere s.t. $M-S$ has two componnents M_1' and M_2', and M_i is obtained by filling in M_i''s boundary sphere with a 3-ball, then M is the <u>connected sum</u> $M_1 \# M_2$

Ex



Connect sum operation is

- well defined (Unique way up to homeo to glue a B^3 to a S^2 boondary component)
- Commutative
- has S^3 as identity

Def A 3-manifold M is prime if $M = P \# Q$ implies $P \cong S^3$ or $Q \cong S^3$

By Alexander's thm S^3 is prime

Def A 3-manifold M is irreducible if every 2-sphere $S \subset M$ bounds a 3-ball in M.

irreducible $\Rightarrow$ prime.

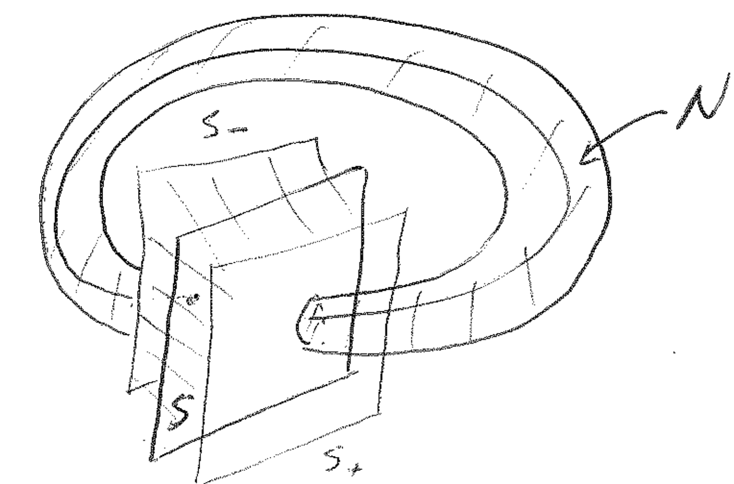
prim $\not\Rightarrow$ irreducible.

Theorem: The only orientable prime 3-manifold which is not irreducible is $S^1 \times S^2$.

Pf| If M is prime and S is a 2-sphere s.t. $M - S$ has 2-components then S bounds a ball. Hence, we can assume $|M - S| = 1$. and $M - S$ is path connected.

Let N be the submanifold consisting of a closed regular nbh of S union an arc from S^+ to S^-.

∂N is a separating 2-sphere in M, so ∂N bounds a 3-ball. Since N is not a 3-ball then $M \cong N \cup_{S^2} B^3 \cong S^2 \times S^1$. □



$= S^2 \times I \cup S^2 \times I = S^2 \times S^1$.

$$M = S^2 \times I \cup D_1^2 \times I \cup D_2^2 \times I \text{ s.t.}$$
$$\partial D_1^2 \times \{t\} = \partial D_2^2 \times \{t\}$$

It remains to show $S^1 \times S^2$ is prime.

Suppose $S^1 \times S^2 \cong V \# W$.

Then $\mathbb{Z} \cong S^1 \times S^2 \cong \pi_1(V) * \pi_1(W)$

WLOG V is simply connected

So V lifts to $\check{V}$ a homeomorphic copy of itself in the universal cover of $S^1 \times S^2$, $\mathbb{R}^3 - \{0\}$.

$\partial \check{V}$ bounds a ball in $\mathbb{R}^3$ by Alexander's thm.

So $\check{V}$ is a ball. Thus $S^1 \times S^2$ is prime.

Prime decomposition theorem

Let M be a compact, connected, orientable 3-manifold. Then there is a decomposition $M = P_1 \# \cdots \# P_n$ with each P_i prime and this decomposition is unique up to insertion or deletion of S^3's.

Existance:

Step 1: Take care of $S^2 \times S^1$ summands.

If M contains non separating spheres, these give rise to $S^2 \times S^1$ summands, as previously seen. Each $S^2 \times S^1$ ~~gives~~ summand contributes $\mathbb{Z}$ to $H_1(M)$, so there must be finitly many.

~~Step 2: We may assume every 2-sphere in M separates.~~

Step 2: Take Care of S^2 boundary components

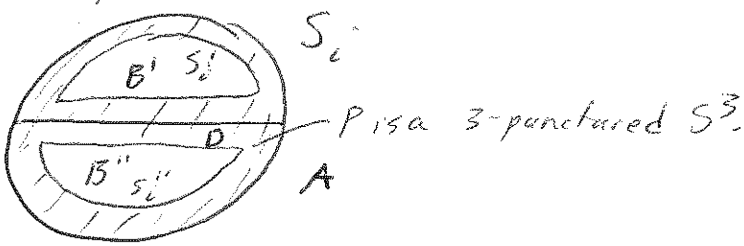
- Each such component corresponds to a B^3 summand and there are finitly many ~~since~~ since M is compact.

Step 3: We may assume ~~M~~ every 2-sphere in M separates and M has no 2-sphere boundary Components.

To complete the proof of existance is safices to show that there is an upper bound on a system of spheres S saticefying

* No component of ~~M-S~~ $M - S$ is a punctured 3-sphere.

Note: If S satisfies $*$ and $S_i \subset S$ s.t. S_i' and S_i'' are obtained by compressing S_i along a disk D, then one of the systems resulting from replacing S_i with S_i' or S_i'' has property $*$.



- If B' and B'' were punctured spheres then $B' \cup B'' \cup P$ is a punctured sphere $*$

- ~~If B' is a punctured sphere,~~
  Suppose B' is not a punctured sphere.
  If $A \cup B'' \cup P$ is a punctured sphere then A is a punctured sphere $\cancel{*}$.

So, one of the two new systems has property $*$.