MATH 240 FINAL EXAM, SPRING 2011

NAME (PRINTED):

TA:

RECITATION TIME:

This examination consists of ten (10 problems). Please *turn off all electronic devices.* You may use both sides of a 8.5×11 sheet of paper for notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**, even on multiple choice or short answer questions—the grading will be bases on your work shown as well as the end result. Please **fill in your final answer in the underlined space** in each problem. Remember to put your name at the top of this page. Good luck.

My signature below certifies that I have complied with the University of Pennsylvania's *code of academic integrity* in completing this examination.

____

Your signature

Problem	**Score (out of)**
1	(10)
2	(10)
3	(10)
4	(10)
5	(10)
6	(10)
7	(10)
8	(10)
9	(10)
10	(10)
Total	(100)

1. (10 pts) (a) Give an example of a 3×3 matrix A which has only two distinct eigenvalues and A is *not* diagonalizable. In other words, there does not exist an invertible 3×3 matrix C such that $C^{-1} \cdot A \cdot C$ is a diagonal matrix. **Justify your answer.**

$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Find the eigenvalues of A

Since A is upper triangular, eigen values ar 0 and 1 (mult 2)

Find the eigenvectors of A

$\lambda = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y + z = 0, \quad z = 0$

So $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 1$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x = 0, \quad z = 0$

So, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Since A only has 2 L.I. eigen vectors it is <u>not</u> diagonalizable.

(b) Give an example of a 3×3 matrix B which has only two distinct eigenvalues and B is diagonalizable. **Justify your answer.**

$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since B is upper triangular, the eigenvalues are $\lambda = 0$ or 1.

Since B is diagonal, B is trivialy diagonalizable:

i.e. $B = I^{-1} B I$.

2. (10 pts) Find A^{201} if

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$$

$A^{201} = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}$.

Solution 1: $A^2 = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So $A^{201} = (A^2)^{100} \cdot A = I^{100} A = I \cdot A = A = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}$

Solution 2: Find P & D s.t. $P^{-1}AP = D$

Step 1: Find eigenvalues $\begin{vmatrix} 3-\lambda & -2 \\ 4 & -3-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(-3-\lambda)+8=0$

$-9+\lambda^2+8=0$

$\lambda^2-1=0$

$(\lambda+1)(\lambda-1)=0$

$\lambda = \pm 1$

Step 2: Find eigen vectors

$\lambda = 1$

$\begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x - y = 0$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -1$

$\begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$4x - 2y = 0$

$x = \frac{1}{2}y$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}y \\ y \end{bmatrix} = y\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$P^{-1} = \frac{1}{1-\frac{1}{2}}\begin{bmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$

$A^{201} = (PDP^{-1})^{201} = PD^{201}P^{-1} = P\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{201}P^{-1} = P\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}P^{-1} = A = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}$

3. (10 pts) Give an example of a homogeneous linear ordinary differential equation which is *not ordinary* at $x = 0$ and has a regular singular point at $x = 0$, which has two linearly *independent* solutions of the form

$$x^{\mu_i} \cdot \left(1 + \sum_{m \geq 1} a_m x^m\right), \quad i = 1, 2, \ \mu_1, \mu_2 \in \mathbb{C}, \ \mu_1 \neq \mu_2$$

for two distinct complex numbers μ_1 and μ_2. **Solve the differential equation you create.** (Hint: The two numbers μ_1, μ_2 in some of the easier examples are integers.)

The differential equation is $x^2 y'' - x y' - 3y = 0$.

The two linearly independent solutions are x^{-1}, x^3.

Take the easy money!

$x^2 y'' - x y' - 3y = 0$ is a cauchy-euler equation with reg. sing. pt at $x = 0$.

Sub in $y = x^m$

$m(m-1)x^m - m x^m - 3x^m = 0$

$m^2 - m - m - 3 = 0$

$m^2 - 2m - 3 = 0$ $\quad m = -1$ or 3

$(m+1)(m-3) = 0$

4. (10 pts) Find the general solution to the following differential equation

$$y'' - 5y' + 6y = e^{2x}$$

(Your answer should involve some unspecified constants.)

$y =$ $C_1 e^{2x} + C_2 e^{3x} - x e^{2x}$.

Find Y_h $\quad y'' - 5y' + 6y = 0$

aux. eq. $m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$m = 2$ or 3

$Y_h = C_1 e^{2x} + C_2 e^{3x}$

Find Y_p : Since Ae^{2x} repeats a homogeneous sol., must guess

$Y_p = Axe^{2x}$

$Y_p' = Ae^{2x} + 2Axe^{2x}$

$Y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} = 4Ae^{2x} + 4Axe^{2x}$

Subing-in $4Ae^{2x} + \cancel{4Axe^{2x}} - 5Ae^{2x} - \cancel{10Axe^{2x}} + \cancel{6Axe^{2x}} = e^{2x}$

$-Ae^{2x} = e^{2x}$

$A = -1$

$Y_g = C_1 e^{2x} + C_2 e^{3x} - xe^{2x}$

5. (10 pts) Let C_1 and C_2 be the closed curves

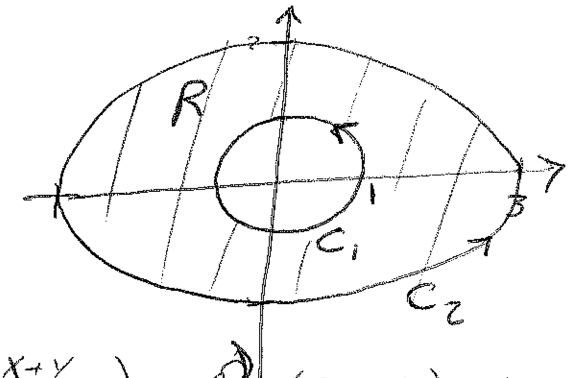
$$C_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}, \quad C_2 = \{(x, y) \in \mathbb{R}^2 \mid 4x^2 + 9y^2 = 36\}$$

on the (x, y)-plane, oriented counterclockwise. Consider the line integrals

$$\oint_{C_i} \frac{(x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y}{x^2+y^2}, \quad i = 1, 2\,.$$

(a) Are the two integrals $\oint_{C_1} \frac{(x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y}{x^2+y^2}$ and $\oint_{C_2} \frac{(x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y}{x^2+y^2}$ equal? Why? (**Justify your answer**.)

Since $\frac{x-y}{x^2+y^2}$ and $\frac{x+y}{x^2+y^2}$ are continuous with continuous derivatives on the region R, then we can apply Green's theorem.



$$\int_{C_2 \cup -C_1} \frac{x-y}{x^2+y^2}\,dx + \frac{x+y}{x^2+y^2}\,dy = \iint_R \frac{\partial}{\partial x}\left(\frac{x+y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x-y}{x^2+y^2}\right) dx\,dy$$

$$= \iint_R \frac{(x^2+y^2)(1) - (x+y)(2x)}{(x^2+y^2)^2} - \frac{(x^2+y^2)(-1) - (x-y)(2y)}{(x^2+y^2)^2}\,dx\,dy$$

$$= \iint_R \frac{-x^2 - 2xy + y^2 - (-x^2 + y^2 - 2xy)}{(x^2+y^2)^2}\,dx\,dy = \iint_R 0\,dx\,dy = 0$$

Since $\int_{C_2 \cup -C_1} \frac{(x-y)dx + (x+y)dy}{x^2+y^2} = 0$, then

$$\int_{C_1} \frac{(x-y)\,dx + (x+y)\,dy}{x^2+y^2} = \int_{C_2} \frac{(x-y)dx + (x+y)\,dy}{x^2+y^2}$$

(b) Evaluate these two line integrals.

$$\oint_{C_1} \frac{(x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y}{x^2+y^2} = \underline{\quad 2\pi \quad}$$

$$\oint_{C_2} \frac{(x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y}{x^2+y^2} = \underline{\quad 2\pi \quad}$$

From part a), it suffices to find $\int_{C_1} \frac{(x-y)\,dx + (x+y)\,dy}{x^2+y^2}$

Let $C_1 : \langle \cos(t), \sin(t) \rangle \quad 0 \le t \le 2\pi$

$$\int_{C_1} \frac{(x-y)\,dx + (x+y)\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{(\cos(t)-\sin(t))(-\sin(t)) + (\cos t + \sin t)(\cos(t))}{\cos^2(t) + \sin^2 t}\,dt$$

$$= \int_0^{2\pi} -\cos(t)\sin(t) + \sin^2(t) + \cos^2(t) + \cos(t)\sin(t)\,dt$$

$$= \int_0^{2\pi} 1\,dt = 2\pi$$

6. (10 pts) Let $S = \partial D$ be the boundary of the solid region D contained in the cylinder $x^2 + y^2 = 4$ between $z = x$ and $z = 8$, i.e.

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 4, \ x \leq z \leq 8\}.$$

Let $\mathbf{n}$ be the unit normal vector field on S pointing outward relative to D. Calculate the flux

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S$$

of the vector field

$$\mathbf{F} = \langle x, y^2, z + y \rangle = x\vec{\imath} + y^2\vec{\jmath} + (z + y)\vec{k}.$$

$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S =$ 64π.

Since all component functions of F and their derivatives are continuous, ~~the~~ and S is the boundary of a solid region, we can use the divergence thm.

$$\iint_S F \cdot n \, dS = \iiint_D \operatorname{div}(F)\, dV = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{x}^{8} 1 + 2y + 1 \, dz \, dx \, dy$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2+2y)(8-x) \, dx \, dy$$

$$= \int_0^{2\pi} \int_0^2 (2 + 2r\sin\theta)(8 - 2r\cos\theta)\, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 16r + 16r^2 \sin\theta - 4r^2\cos\theta - 4r^3 \sin\theta\cos\theta \, dr \, d\theta$$

$$= \int_0^{2\pi} 32 + \underline{\frac{128}{3}\sin\theta - \frac{32}{3}\cos\theta - 16\sin\theta\cos\theta} \, d\theta$$

these are periodic on $[0, 2\pi]$

$$= \boxed{64\pi}$$

7. (10 pts) Find the general solution to the following system of differential equations

$$\frac{d}{dt}\mathbf{X} = \begin{pmatrix} 1 & -8 \\ 1 & -3 \end{pmatrix}\mathbf{X}, \qquad \text{where } \mathbf{X} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

(Your answer should involve some unspecified constants.)

$\mathbf{X} =$ ____.

Step 1: Find the eigen values of $\begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & -8 \\ 1 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda)+8 = -3+2\lambda+\lambda^2+8 = \lambda^2+2\lambda+5=0$$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2\pm 4i}{2} = -1 \pm 2i$$

Step 2: Find eigen values

$$\begin{bmatrix} 1-(-1-2i) & -8 \\ 1 & -3-(-1-2i) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2+2i & -8 \\ 1 & -2+2i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, $(2+2i)x - 8y = 0 \quad y = \left(\frac{1}{4}+\frac{1}{4}i\right)x$

$\cancel{x = \frac{8}{2+2i}}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ \frac{1}{4}+\frac{1}{4}i \end{bmatrix} \text{ e.vec.}$$

From our formula

$$X = c_1\left(\begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}\sin(2t)\right)e^{-t}$$

$$+ c_2\left(\begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}\cos(2t) + \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}\sin(2t)\right)e^{-t}$$

8. (10 pts) Find a recursion formula for the coefficients a_n's of a power series expansion of a function

$$y(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$$

defined on $(-1, 1)$ which satisfies the following differential equation

$$(x-1)\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$$

Recursion formula: $C_{k+2} = \dfrac{C_{k+1}(k+1)}{k+2}$.

In other words find a solution $y = \sum_{n=0}^{\infty} C_n x^n$ where $C_0 = 1$

$$(x-1)\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} C_n n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} C_n (n)(n-1) x^{n-1} - \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2} + \sum_{n=1}^{\infty} C_n n x^{n-1} = 0$$

$k = n-1$, $n = k+1$ $\qquad$ $k = n-2$, $n = k+2$ $\qquad$ $k = n-1$, $n = k+1$

$$\sum_{k=1}^{\infty} C_{k+1}(k+1)(k)x^k - \sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k + \sum_{k=0}^{\infty} C_{k+1}(k+1)x^k = 0$$

$$(-C_2(2) + C_1) + \sum_{k=1}^{\infty}\left[C_{k+1}(k+1)^2 - C_{k+2}(k+2)(k+1)\right]x^k = 0$$

$$C_{k+2} = \frac{C_{k+1}(k+1)}{(k+2)}$$

9. (10 pts) Find the general solution to the system of linear ordinary differential equation

$$\frac{d}{dx}u(x) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} u(x), \quad \text{where} \quad u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \end{pmatrix}$$

(Your answer should involve some unspecified constants.)

$u(x) =$ ____

- Step 1: Find eigenvalues

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda)(1-\lambda)$$

$$\lambda = 2 \text{ or } \underline{1}$$

- Step 2: Find eigenvectors

$\lambda = 2$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y = 0 \quad x = z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y = 0, \ x = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} z$$

Must find P s.t. $(A - 2I)P = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore \ P_2 = 1 \qquad P_1 - P_3 = 1 \qquad P = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

Pick $P_1 = 2$ and $P_3 = 1$

Putting it all together:

$$c_1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{2t} + c_3\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}te^{2t} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}e^{2t}\right)$$

10. (10 pts) Mark the each of the following following five (5) statements true or false. You do **not** need to justify your answer.

F A. Suppose A is a 4×4 matrix and a is a real number such that a, $a+1$, $a+2$ and $2a+3$ are the eigenvalues of A; in other words the characteristic polynomial of A is

$$\det(\lambda \cdot I_4 - A) = (\lambda - a)(\lambda - a - 1)(\lambda - a - 2)(\lambda - 2a - 3).$$

Then A is diagonalizable.

$$a = 2a + 3$$
$$-a = 3$$
$$a = -3$$

$$\begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$ is <u>not</u> diagonalizeable

T B. For every 2×2 matrix A of rank 1 there exists a 2×2 matrix B of rank 1 such that $A \cdot B = 0$, where 0 is the 2×2 matrix with all zero entries.

$$A = \begin{bmatrix} a & b \\ ca & cb \end{bmatrix} \qquad B = \begin{bmatrix} x & y \\ zx & zy \end{bmatrix}$$

$$AB = \begin{bmatrix} a & b \\ ca & cb \end{bmatrix} \begin{bmatrix} x & y \\ zx & zy \end{bmatrix} = \begin{bmatrix} ax + bzx & ay + bzy \\ cax + cbzx & cay + cbzy \end{bmatrix}$$

these entries are all zero if

$$a + bz = 0$$
$$z = \frac{-a}{b}$$

T C. The differential equation $x\frac{d^2y}{dx^2} + (\cos(x) - 1)\frac{dy}{dx} = 0$ for the function $y(x)$ on $\mathbb{R}$ is equivalent to a linear ordinary differential equation for $y(x)$ on $\mathbb{R}$ which has *no singular point.*

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0$$

F D. Suppose that $f_0(x)$, $f_1(x)$, and $f_2(x)$ are polynomials in x, and $f_2(x)$ is *not* identically 0. If $y_1(x)$ and $y_2(x)$ are smooth functions (i.e. they can be differentiated infinitely many times) on $\mathbb{R}$ such that $y_1(0) = y_2(0)$, $y_1'(0) = y_2'(0)$, and

$$f_2(x)y_i'' + f_1(x)y_i' + f_0(x)y_i = 0 \quad \text{for } i = 1, 2,$$

then $y_1(x) = y_2(x)$.

$y = 0$ and $y = x^2$ are solutions to the IVP

$$x^2 y'' - x y' = 0$$

T E. Suppose that A is a 2×2 matrix with real entries, and $x(t), y(t)$ are two differentiable functions on $\mathbb{R}$ such that

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \text{and} \quad x(t)^2 + y(t)^2 = 1$$

for all real numbers $t \in \mathbb{R}$. Then the two eigenvalues λ_1, λ_2 of A has both purely imaginary numbers, i.e. $\lambda_1, \lambda_2 \in \sqrt{-1} \cdot \mathbb{R}$.

The only way to get closed loops in the phase portrait is if λ_1 and λ_2 are purely imaginary.