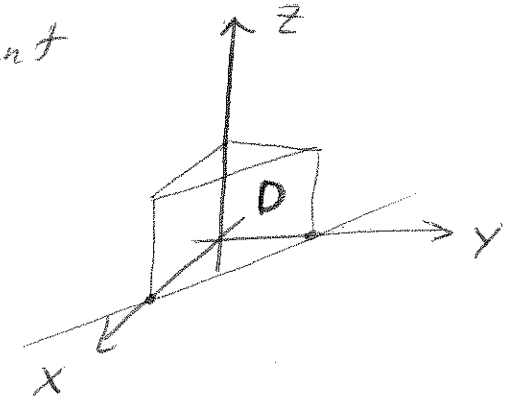
1. If $\mathbf{F} = \langle xy, y^2z, z^3 \rangle$, evaluate $\int\int_S (\mathbf{F}\cdot\mathbf{n})dS$ where S is the outward oriented boundary of the region bounded by $z=0$, $z=1$, $y=1-x$, $x=0$ and $y=0$.

Since F has polynomial component functions we can use the divergence theorem



Step 1: Find $\operatorname{div}(F) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \circ \langle xy, y^2z, z^3 \rangle$

$$= y + 2yz + 3z^2$$

Step 2: Find bounds on $\iiint \operatorname{div}(F)\, dV$

$$\iint_S F\circ\vec{n}\, dS = \int_0^1 \int_0^{1-x} \int_0^1 y + 2yz + 3z^2 \, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} yz + yz^2 + z^3 \Big|_{z=0}^{z=1} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} y + y + 1 \, dy\, dx = \int_0^1 y^2 + y \Big|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 1 - 2x + x^2 + 1 - x \, dx$$

$$= \frac{x^3}{3} - \frac{3}{2}x^2 + 2x \Big|_0^1 = \frac{1}{3} - \frac{3}{2} + 2 = \boxed{\frac{5}{6}}$$

2. Derive Green's Theorem using Stokes' Theorem.

Let C be a piecewise smooth simple closed curve bounding a **region** R in the xy-plane.

Let $F = P(x,y)\vec{i} + Q(x,y)\vec{j}$ be a vec. field in the plane s.t. $P, Q, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are continuous.

Let $F^* = P(x,y)\vec{i} + Q(x,y)\vec{j} + 0\vec{k}$ a vec. field in $\mathbb{R}^3$.

Since $P, Q, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are continuous, then the component functions and all of the partial derivatives of the component functions of F^* are continuous. $\}$ $*$

Examine $\oint_C F^* \circ dr = \oint_C \langle P, Q, 0\rangle \circ \langle dx, dy, dz\rangle = \oint_C P\,dx + Q\,dy$

Examine the surface integral

$$\iint_R \operatorname{curl}(F^*) \circ n \, dA = \iint_R \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle \circ \langle 0,0,1\rangle \, dA$$

$$= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

By $*$, Stokes theorem holds, so $\oint_C F^* \circ dr = \iint_R \operatorname{curl}(F^*) \circ n \, dA$

Thus, $\oint_C P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$ $\square$

3. Let C_1 and C_2 be the closed curves

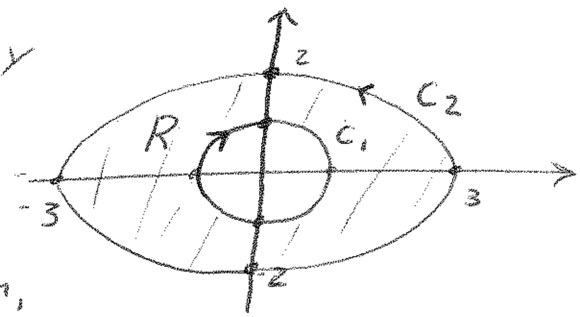
$$C_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}, \quad C_2 = \{(x,y) \in \mathbb{R}^2 \mid 4x^2 + 9y^2 = 36\}$$

on the (x,y)-plane, oriented counterclockwise. Consider the line integrals

$$\oint_{C_i} \frac{(x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y}{x^2+y^2}, \quad i = 1, 2.$$

(a) Are the two integrals $\oint_{C_1} \frac{(x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y}{x^2+y^2}$ and $\oint_{C_2} \frac{(x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y}{x^2+y^2}$ equal? Why? (**Justify your answer.**)

Since the region R has boundary $C_2 \cup -C_1$ and the vector field in question meets the hypotheses of Green's theorem, we can apply Green's theorem to R to get:



$$\oint_{C_2 \cup -C_1} \frac{x-y}{x^2+y^2}\,dx + \frac{x+y}{x^2+y^2}\,dy = \iint_R \frac{\partial}{\partial x}\left(\frac{x+y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x-y}{x^2+y^2}\right) dA$$

$$= \iint_R \frac{y^2-2xy-x^2}{(x^2+y^2)^2} - \frac{y^2-2xy+x^2}{(x^2+y^2)^2}\,dA = 0$$

Thus, $\oint_{C_2} \frac{x-y}{x^2+y^2}\,dx + \frac{x+y}{x^2+y^2}\,dy + \oint_{-C_1} \frac{x-y}{x^2+y^2}\,dx + \frac{x+y}{x^2+y^2}\,dy = 0$

or

$$\boxed{\oint_{C_2} \frac{x-y}{x^2+y^2}\,dx + \frac{x+y}{x^2+y^2}\,dy = \oint_{C_1} \frac{x-y}{x^2+y^2}\,dx + \frac{x+y}{x^2+y^2}\,dy}$$

(b) Evaluate these two line integrals.

$$\oint_{C_1} \frac{(x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y}{x^2+y^2} = \underline{\quad 2\pi \quad}$$

$$\oint_{C_2} \frac{(x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y}{x^2+y^2} = \underline{\quad 2\pi \quad}$$

C_1 : $x(t) = \cos(t)$ $y(t) = \sin(t)$ $0 \le t \le 2\pi$

$$\oint_{C_1} \frac{x-y}{x^2+y^2}\,dx + \frac{x+y}{x^2+y^2}\,dy = \int_0^{2\pi} \frac{\cos(t)-\sin(t)}{\cos^2(t)+\sin^2(t)}(-\sin(t))dt + \frac{\cos(t)+\sin(t)}{\cos^2(t)+\sin^2(t)}\cos(t)dt$$

$$= \int_0^{2\pi} -\sin(t)\cos(t) + \sin^2(t) + \cos^2(t) + \sin(t)\cos(t)\,dt$$

$$= \int_0^{2\pi} dt = \boxed{2\pi}$$

The above two integrals are equal by part a)

4.

a) Find a primitive of the differential $e^{2z}dx + 3y^2dy + 2xe^{2z}dz$.

$$\phi(x, y, z) = xe^{2z} + y^3$$

$$d\phi = \frac{\partial}{\partial x}(xe^{2z} + y^3)\,dx + \frac{\partial}{\partial y}(xe^{2z} + y^3)\,dy + \frac{\partial}{\partial z}(xe^{2z} + y^3)\,dz$$

$$d\phi = e^{2z}dx + 3y^2dy + 2xe^{2z}dz$$

b) Use the answer from part a) to evaluate the following line integral

$$\int_{(1,1,ln(3))}^{(2,2,ln(3))} e^{2z}dx + 3y^2dy + 2xe^{2z}dz$$

$$= \phi(2, 2, \ln(3)) - \phi(1, 1, \ln(3))$$

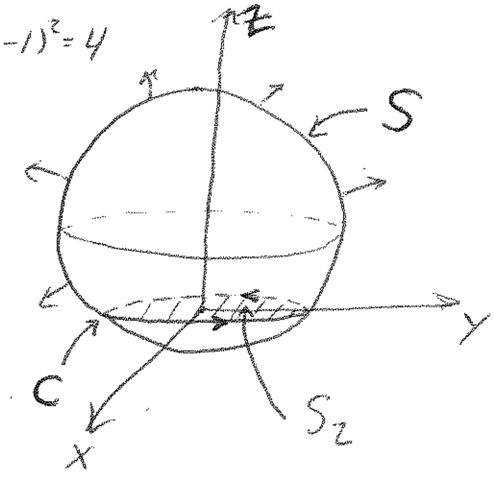
$$= 2e^{2\ln(3)} + 2^3 - (e^{2\ln(3)} + 1^3)$$

$$= 2 \cdot 9 + 8 - (9 + 1)$$

$$= \boxed{16}$$

5. Let S be the portion of the outward oriented sphere $x^2 + y^2 + (z-1)^2 = 4$ above the plane $z = 0$. Evaluate $\int\int_S (curl(\mathbf{F}) \cdot \mathbf{n}) dS$ when $\mathbf{F} = < xy^2 + y + e^{z^2}, x^2y + ze^z, xyz >$.

Find the intersection of $z=0$ and $x^2+y^2+(z-1)^2=4$



$$X^2+Y^2+(1)^2=4$$

$$X^2+Y^2=3$$

So, C is the circle in the xy-plane centered at $(0,0)$ of radius $\sqrt{3}$.

Since F has component functions consisting of sums & products of polynomials & exponentials, the hypotheses of Stokes' theorem are met.

By Stokes' $\iint_S curl(F) \circ n \, dS = \oint_C F \circ dr$

Let S_2 be the disk of radius $\sqrt{3}$ in the xy-plane and apply stokes' again

$$\oint_C F \circ dr = \iint_{S^2} curl(F) \circ n \, dS$$

$$= \iint_{S^2} \langle ?, ?, 2xy - (2xy+1) \rangle \circ \langle 0,0,1 \rangle \, dS$$

$$= \iint_{S^2} -1 \, dS = -\pi(\sqrt{3})^2 = \boxed{-3\pi}$$