Math 240: Homogeneous Linear Systems of D.E.s

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Outline

Review of Last Time

1. Defined systems of differential equations
2. Developed the notion of Linear Independence.
3. Developed the notion of General Solution.

Linear systems

Definition

The following is a **first order system**

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n} + f_1(t)$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n} + f_2(t)$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn} + f_n(t)$$

Where each x_i is a function of t.

The Wronskian

Theorem

Let X_1, X_2, ..., X_n be n solution vectors to a homogeneous system on an interval I. They are linearly independent if and only if their **Wronskian** *is non-zero for every t in the interval.*

Today's Goals

1. Be able to solve constant coefficient systems.

Guessing a Solution

Given a constant coefficient, linear, homogeneous, first-order system

$$\mathbf{X'=AX}$$

our intuition prompts us to guess a solution vector of the form

$$\mathbf{X} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = \mathbf{K} e^{\lambda t}$$

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Hence, we can find such a solution vector iff K is an eigenvector for A with eigenvalue λ.

General Solution with Distinct Real Eigenvalues

Theorem

Let λ_1, λ_2, ..., λ_n *be n* **distinct** *real eigenvalues of the* $n \times n$ *coefficient matrix* **A** *of the homogeneous system* **X'=AX**, *and let* $\mathbf{K}_1$, $\mathbf{K}_2$, ... , $\mathbf{K}_n$ *be the corresponding eigenvectors. Then the general solution on* $(-\infty, \infty)$ *is*

$$\mathbf{X} = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} + \ldots + c_n\mathbf{K}_n e^{\lambda_n t}$$

where the c_i *are arbitrary constants.*

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where the c_i are arbitrary constants.

Exercise: Solve the linear system $X' = AX$ if

$$A = \begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}$$

Repeated Eigenvalues

In a $n \times n$ linear system there are two possibilities for an eigenvalue λ of multiplicity 2.

1. λ has two linearly independent eigenvectors $\mathbf{K}_1$ and $\mathbf{K}_2$.
2. λ has a single eigenvector $\mathbf{K}$ associated to it.

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In the first case, there are linearly independent solutions $\mathbf{K}_1 e^{\lambda t}$ and $\mathbf{K}_2 e^{\lambda t}$.

In the second case, there are linearly independent solutions $\mathbf{K} e^{\lambda t}$ and

$$[\mathbf{K} t e^{\lambda t} + \mathbf{P} e^{\lambda t}]$$

where we find $\mathbf{P}$ be solving $(\mathbf{A} - \lambda \mathbf{I})\mathbf{P} = \mathbf{K}$

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where we find **P** be solving $(\mathbf{A} - \lambda \mathbf{I})\mathbf{P} = \mathbf{K}$
Exercise: Solve the linear system $X' = AX$ if

$$A = \begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix}$$

How Bad Can it Get?

In general, you will only be asked to solve systems $X' = AX$ if the multiplicity of the eigenvalues of A is at most 1 more than the number of linearly independent eigenvectors for that value. In this case you need to find at most one vector $\mathbf{P}$ such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{P} = \mathbf{K}$

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Exercise: Solve the linear system $X' = AX$ if

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Real and Imaginary Parts of a Matrix

Given an $n \times m$ matrix A with complex entries,

$Re(A)$ is the real $n \times m$ matrix of the purely real entries in A and

$Im(A)$ is the real $n \times m$ matrix of purely imaginary entries of A.

Complex Eigenvalues

Now we DO have to find eigenvectors for complex eigenvalues

Theorem

Let $\lambda = \alpha + i\beta$ be a complex eigenvalue of the coefficient matrix A in a homogeneous linear system $\mathbf{X}' = \mathbf{A}\ \mathbf{X}$, and $\mathbf{K}$ be the corresponding eigenvector. Then

$$\mathbf{X}_1 = [Re(\mathbf{K})cos(\beta t) - Im(\mathbf{K})sin(\beta t)]e^{\alpha t}$$

$$\mathbf{X}_2 = [Im(\mathbf{K})cos(\beta t) + Re(\mathbf{K})sin(\beta t)]e^{\alpha t}$$

are linearly independent solutions to $\mathbf{X}' = \mathbf{A}\ \mathbf{X}$ on $(-\infty, \infty)$.

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Exercise: Solve the linear system $X' = AX$ if

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$